Envelopes and String Art

Gregory Quenell

Activity:

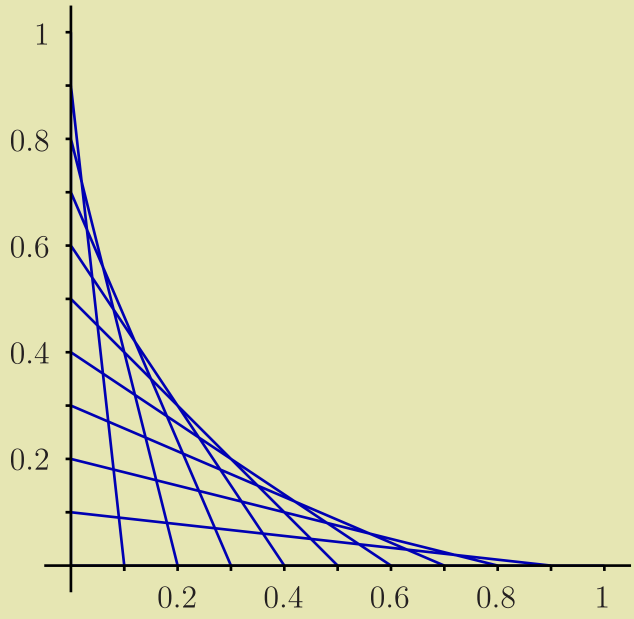
Draw line segments connecting

$$(0, x) \text{ with } (1 - x, 0)$$

for $x = 0.1,\ 0.2,\ \ldots,\ 0.9$.

Benefits:

- Gives you something to do during calculus class
- Makes a pleasing pattern of intersecting lines



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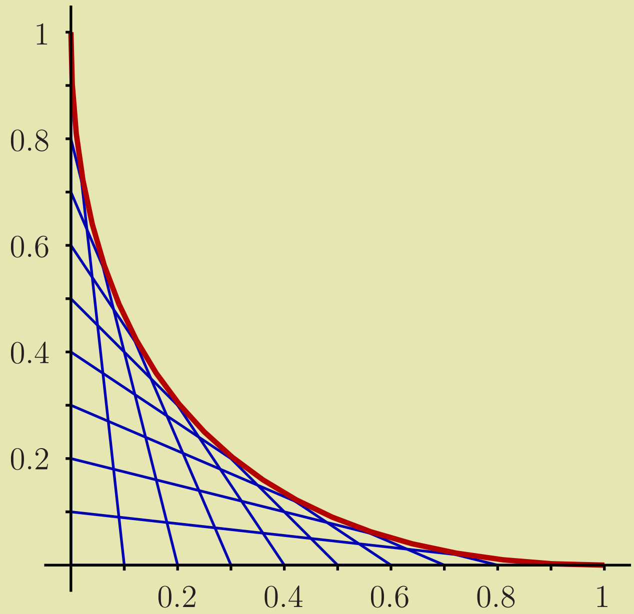
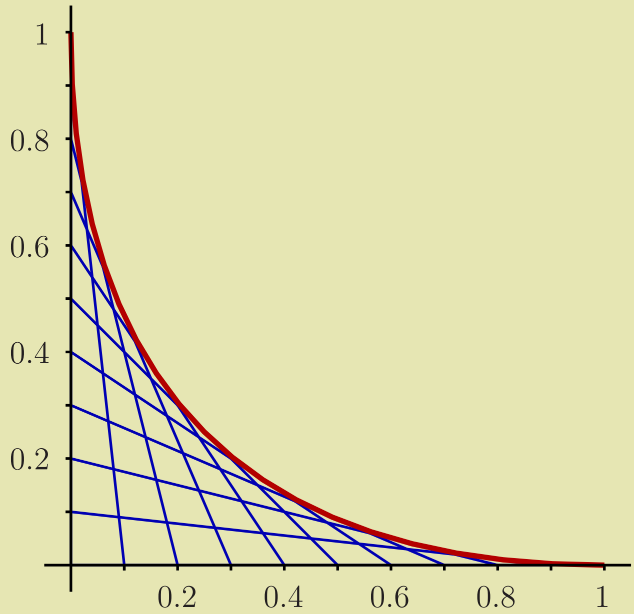


Benefits:

- Gives you something to do during calculus class
- Makes a pleasing pattern of intersecting lines
- Provides an interesting curve to study

Question:

What curve is this?



Observation:

The curve's defining property is that the sum of the x- and y-intercepts of each of its tangent lines is 1.

That gives us the condition

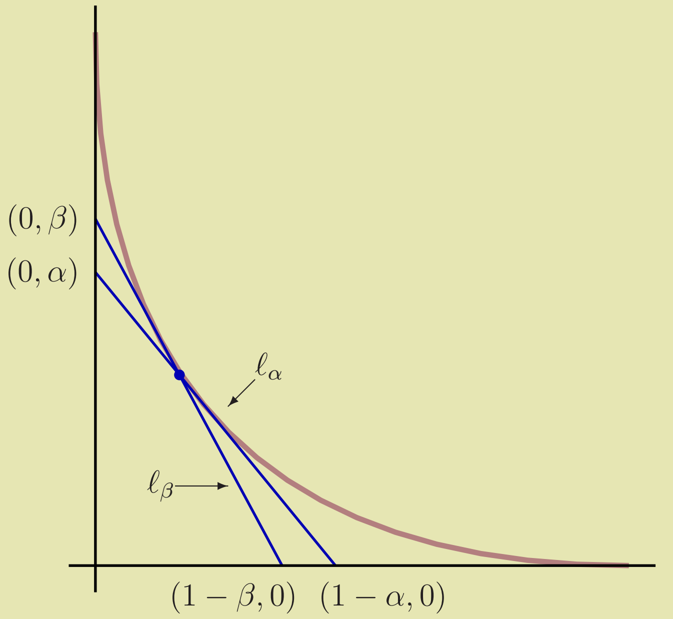
$$y - x\frac{dy}{dx} + x - \frac{y}{dy/dx} = 1$$

Different approach:

For each $\alpha \in [0,1]$, let ℓ_α be the line segment connecting

$$(0, \alpha) \text{ with } (1-\alpha, 0).$$

If α and β are close together, then the intersection point of ℓ_α and ℓ_β is close to a point on the curve.

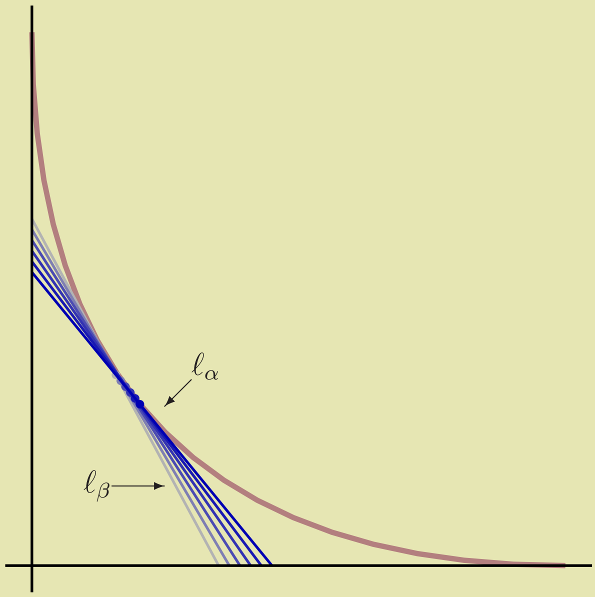


Exercise:

For $\alpha \neq \beta$, the segments ℓ_α and ℓ_β intersect at the point

$$(\alpha\beta, (1-\alpha)(1-\beta)).$$

Result:



As $\beta \to \alpha$, the point

$$(\alpha\beta, (1-\alpha)(1-\beta))$$

approaches a point on the curve. Thus, each point on the curve has the form

$$\lim_{\beta\to\alpha}(\alpha\beta, (1-\alpha)(1-\beta))$$

for some α.

This is an easy limit, and we get the parametrization

$$(\alpha^2, (1-\alpha)^2), \quad 0 \le \alpha \le 1$$

for our envelope curve.

Remarks:

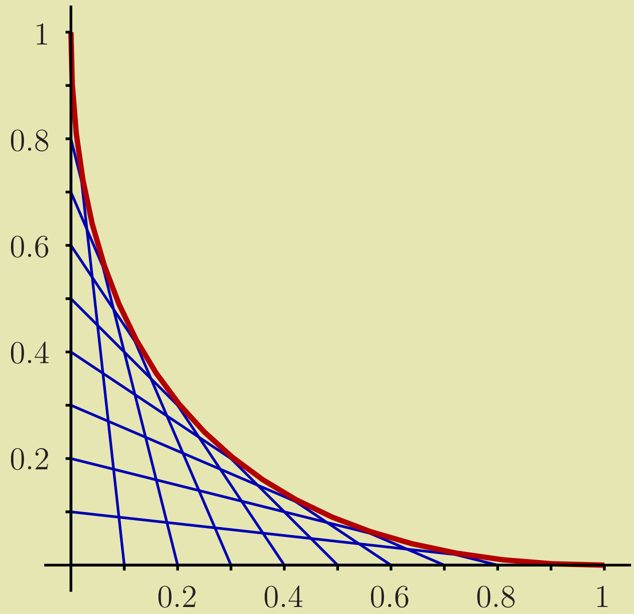
- The coordinates

$$x = \alpha^2 \quad \text{and} \quad y = (1-\alpha)^2$$

satisfy

$$\sqrt{x} + \sqrt{y} = 1$$

so our curve is (one branch of) a *hypocircle* with exponent $\frac{1}{2}$.



- Stewart, p. 234, problem 8 says

  "Show that the sum of the x- and y-intercepts of any tangent line to the curve $\sqrt{x} + \sqrt{y} = \sqrt{c}$ is equal to c."

Exercise:

The coordinates

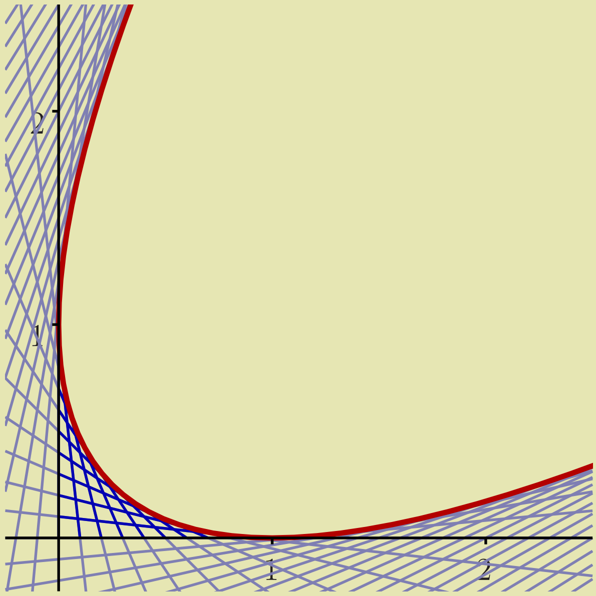
$$x = \alpha^2 \text{ and } y = (1-\alpha)^2$$

satisfy

$$2(x+y) \;=\; (x-y)^2 + 1$$

Result:

Our envelope curve lies on a parabola in the uv-plane, where $u = x + y$ and $v = x - y$.



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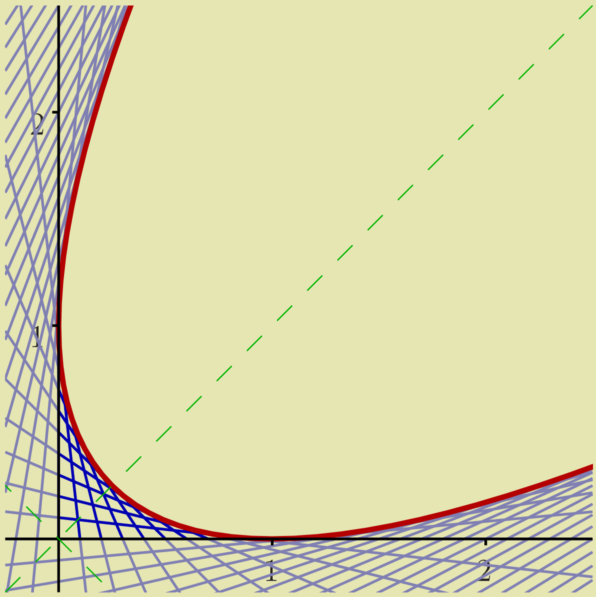
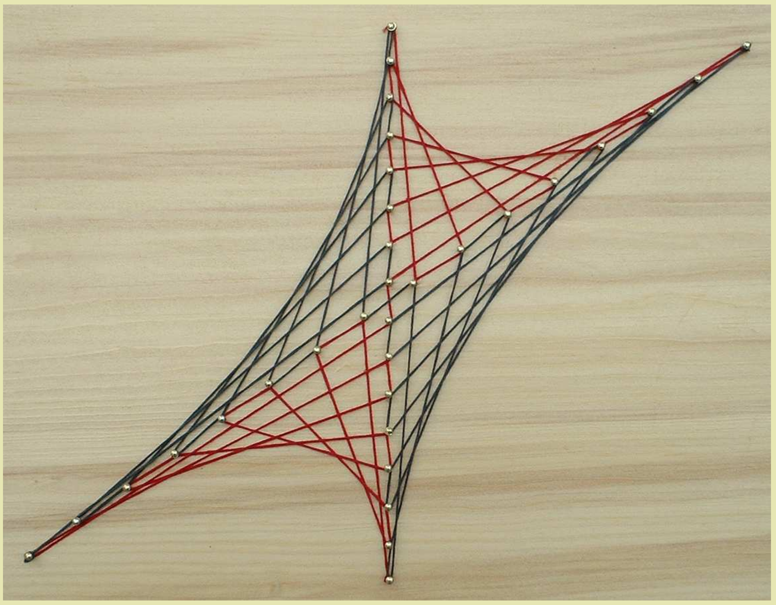
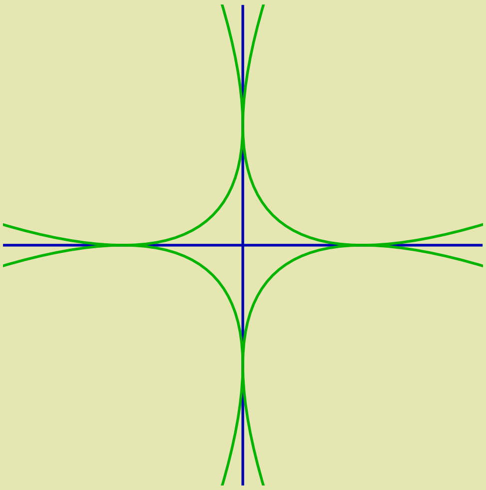


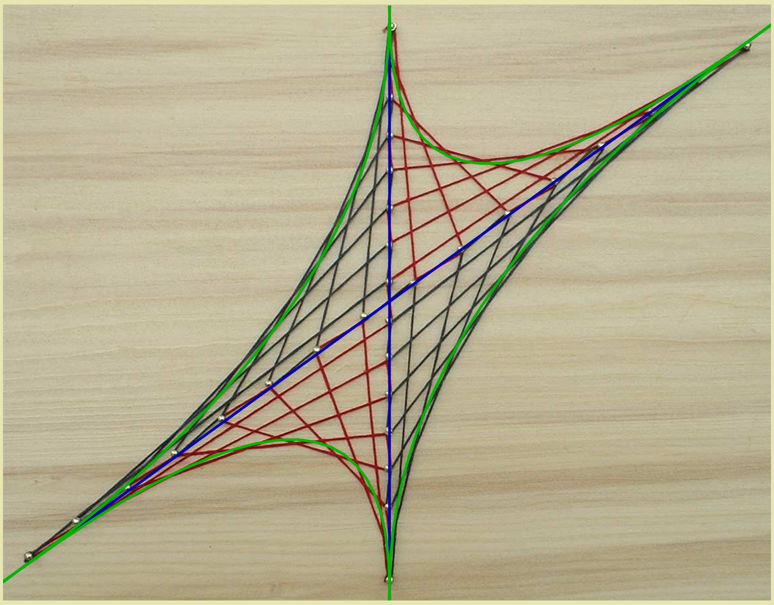
Activity: String Art

Drive nails at equal intervals along two lines, and connect the nails with decorative string.



Activity: String Art





The envelope curves are the images, under a linear transformation, of parabolas tangent to the coordinate axes. That is, they are parabolas tangent to the nailing lines.

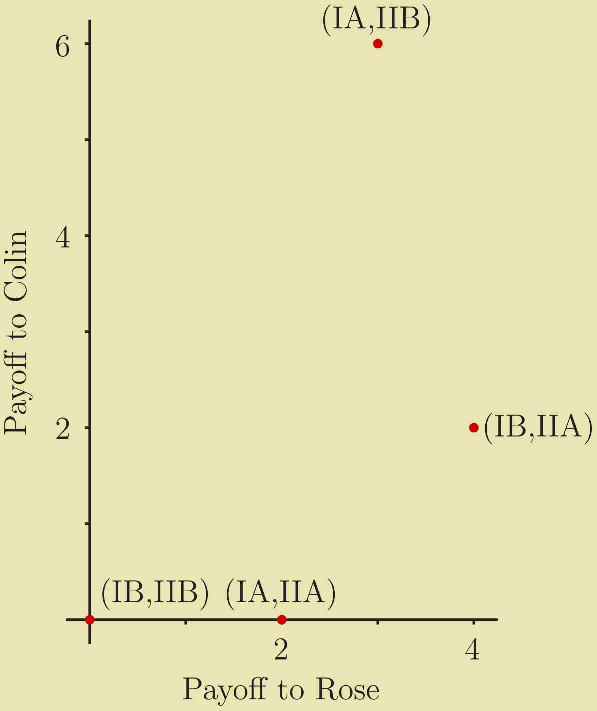
Digression: Game Theory

Consider a two-person, non-zero-sum game in which each player has two strategies.

		Colin	
		IIA	IIB
Rose	IA	$(2, 0)$	$(3, 6)$
	IB	$(4, 2)$	$(0, 0)$

Such a game has four possible payoffs. We list them in a *payoff matrix*.

We can show the payoffs to Rose and Colin as points in the *payoff plane*.



Assumptions:

We assume each player adopts a *mixed strategy*:

- Rose plays IA with probability p and IB with probability $1 - p$.
- Colin plays IIA with probability q and IIB with probability $1 - q$

		Colin IIA	Colin IIB
Rose	IA	$(2,0)$	$(3,6)$
	IB	$(4,2)$	$(0,0)$

The *expected payoff* is then

$$pq(2,0) + p(1-q)(3,6) + (1-p)q(4,2) + (1-p)(1-q)(0,0)$$

or

$$p\left[q(2,0) + (1-q)(3,6)\right] + (1-p)\left[q(4,2) + (1-q)(0,0)\right]$$

or

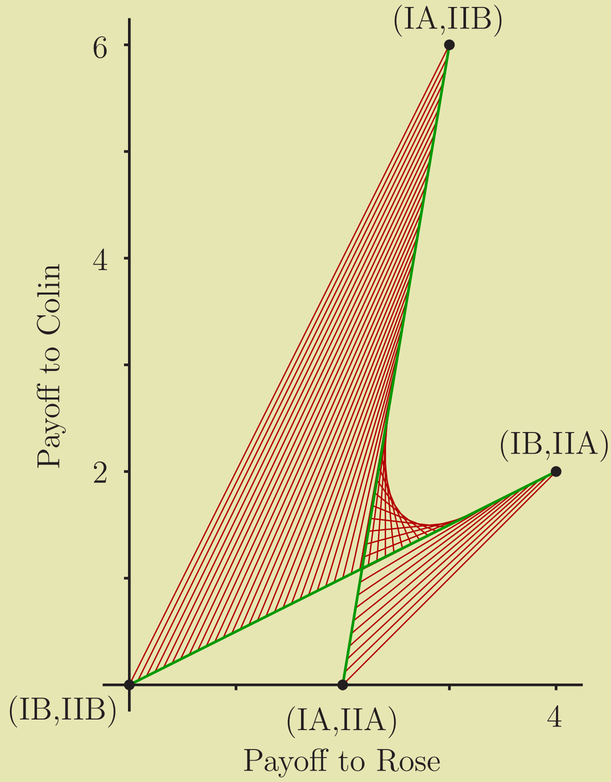
$$q\left[p(2,0) + (1-p)(4,2)\right] + (1-q)\left[p(3,6) + (1-p)(0,0)\right]$$

Possible payoff points:

Each value of q determines one point on the line from $(2, 0)$ to $(3, 6)$ and one point on the line from $(4, 2)$ to $(0, 0)$.

Then p is the parameter for a line segment between these points.

$$p\left[q(2,0)+(1-q)(3,6)\right] \\ +(1-p)\left[q(4,2)+(1-q)(0,0)\right]$$

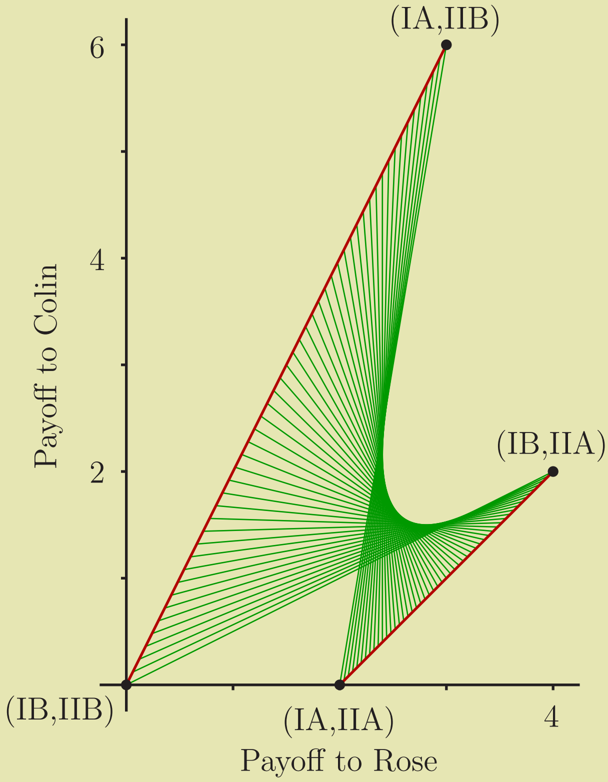


Possible payoff points:

Alternatively, each value of p determines one point on the line from $(2,0)$ to $(4,2)$ and one point on the line from $(3,6)$ to $(0,0)$.

Then q is the parameter for a line segment between these points.

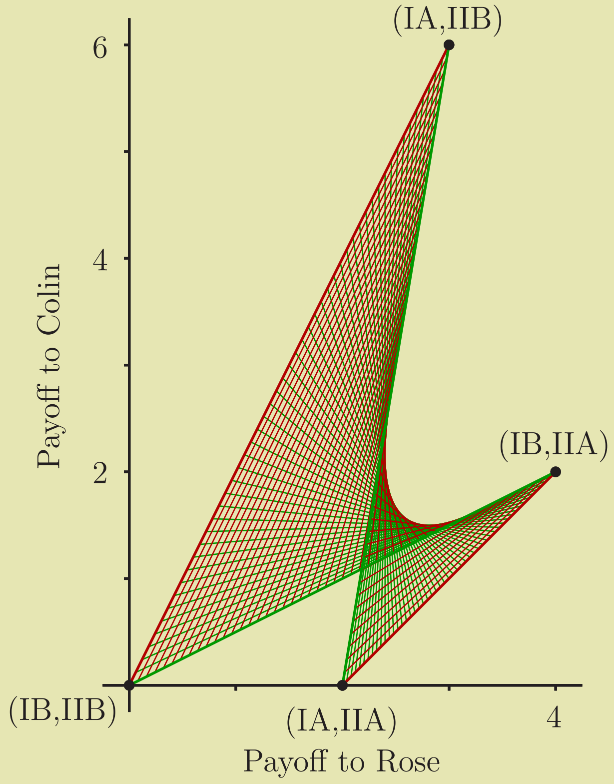
$$q\,[p(2,0)+(1-p)(4,2)] \\ +(1-q)\,[p(3,6)+(1-p)(0,0)]$$



Possible payoff points:

Either way, the expected payoff is contained in a region bounded by four lines and a parabolic envelope curve.

If the game is played a large number of times and the average payoff converges to a point outside this region, then the players' randomizing devices are not independent.



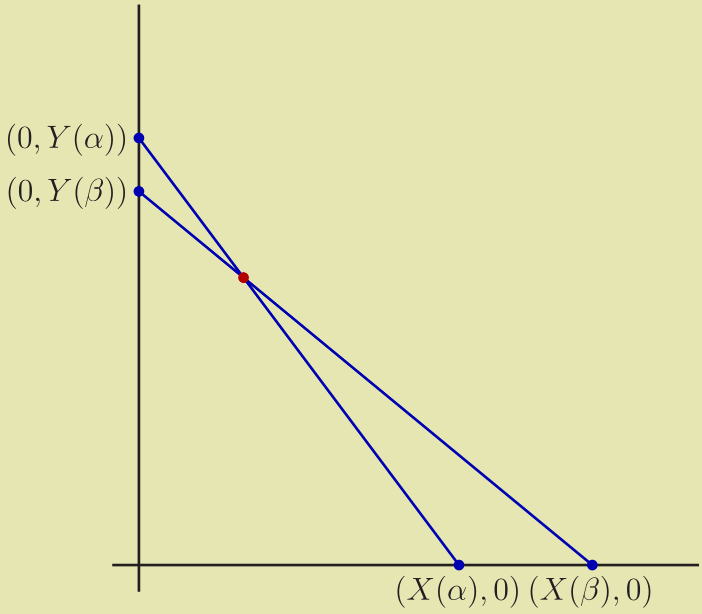
This could be due to collusion, espionage, or maybe just poor random-number generators.

Generalization:
Unequal Spacing

Draw line segments ℓ_α connecting

$$(X(\alpha), 0) \;\text{ with }\; (0, Y(\alpha))$$

for arbitrary differentiable functions X and Y.
These are "spacing functions".

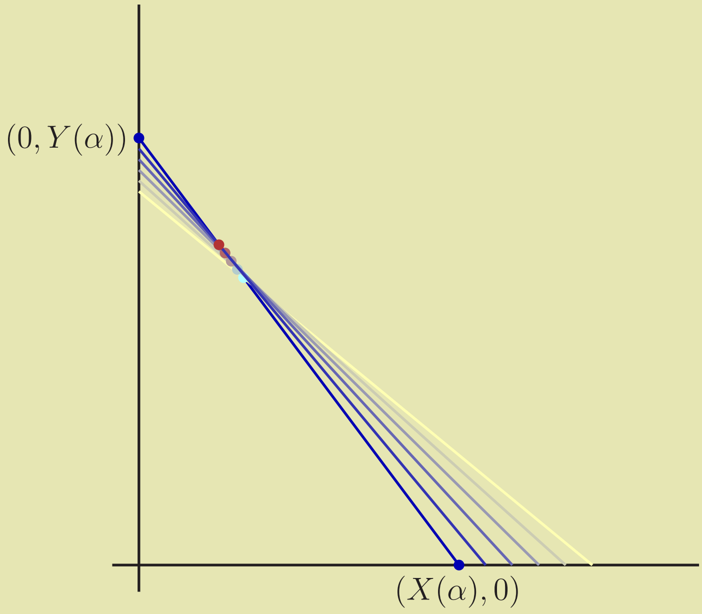


Exercise:

Segments ℓ_α and ℓ_β intersect at the point

$$\left(\frac{X(\alpha)X(\beta)(Y(\beta) - Y(\alpha))}{X(\alpha)Y(\beta) - Y(\alpha)X(\beta)}, \frac{Y(\alpha)Y(\beta)(X(\alpha) - X(\beta))}{X(\alpha)Y(\beta) - Y(\alpha)X(\beta)}\right)$$

Generalization: Unequal Spacing



To find a point on the envelope curve, we need to compute

$$\lim_{\beta \to \alpha} \left(\frac{X(\alpha)X(\beta)(Y(\beta) - Y(\alpha))}{X(\alpha)Y(\beta) - Y(\alpha)X(\beta)}, \frac{Y(\alpha)Y(\beta)(X(\alpha) - X(\beta))}{X(\alpha)Y(\beta) - Y(\alpha)X(\beta)} \right)$$

Calculation:

"Plugging in" α for β gives

$$\left(\frac{X(\alpha)X(\alpha)(Y(\alpha)-Y(\alpha))}{X(\alpha)Y(\alpha)-Y(\alpha)X(\alpha)}, \frac{Y(\alpha)Y(\alpha)(X(\alpha)-X(\alpha))}{X(\alpha)Y(\alpha)-Y(\alpha)X(\alpha)}\right)$$

$$= \left(\frac{0}{0}, \frac{0}{0}\right)$$

So we try something else ...

The x-coordinate of a point on the envelope is

$$\lim_{\beta\to\alpha} \frac{X(\alpha)X(\beta)(Y(\beta)-Y(\alpha))}{X(\alpha)Y(\beta)-Y(\alpha)X(\beta)}$$

Calculation: $\lim_{\beta\to\alpha} \frac{X(\alpha)X(\beta)(Y(\beta)-Y(\alpha))}{X(\alpha)Y(\beta)-Y(\alpha)X(\beta)}$

$$= \lim_{\beta\to\alpha} \frac{X(\alpha)X(\beta)(Y(\beta)-Y(\alpha))}{X(\alpha)Y(\beta)-X(\alpha)Y(\alpha)+X(\alpha)Y(\alpha)-Y(\alpha)X(\beta)}$$

$$= \lim_{\beta\to\alpha} \frac{X(\alpha)X(\beta)(Y(\beta)-Y(\alpha))}{X(\alpha)(Y(\beta)-Y(\alpha))-Y(\alpha)(X(\beta)-X(\alpha))}$$

$$= \lim_{\beta\to\alpha} \frac{X(\alpha)X(\beta)(\frac{Y(\beta)-Y(\alpha)}{\beta-\alpha})}{X(\alpha)(\frac{Y(\beta)-Y(\alpha)}{\beta-\alpha})-Y(\alpha)(\frac{X(\beta)-X(\alpha)}{\beta-\alpha})}$$

$$= \frac{X(\alpha)X(\alpha)\cdot \lim_{\beta\to\alpha}\frac{Y(\beta)-Y(\alpha)}{\beta-\alpha}}{X(\alpha)\cdot \lim_{\beta\to\alpha}\frac{Y(\beta)-Y(\alpha)}{\beta-\alpha}-Y(\alpha)\cdot \lim_{\beta\to\alpha}\frac{X(\beta)-X(\alpha)}{\beta-\alpha}}$$

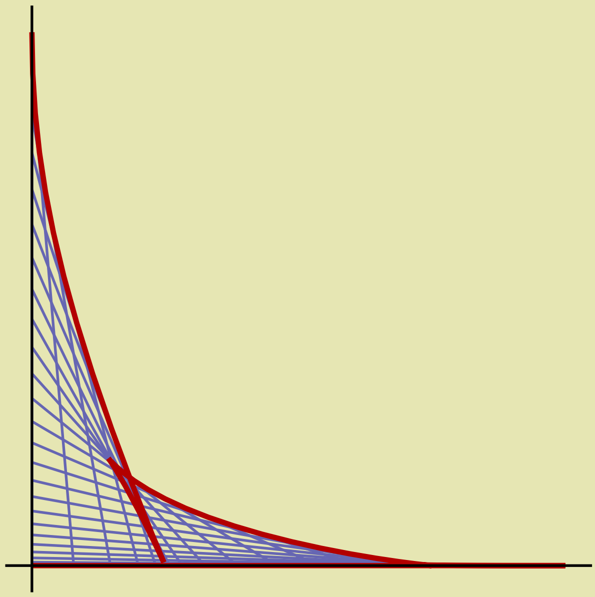
$$= \frac{(X(\alpha))^2Y'(\alpha)}{X(\alpha)Y'(\alpha)-Y(\alpha)X'(\alpha)}$$

Result:

Do the same thing for the y-coordinate

$$\lim_{\beta \to \alpha} \frac{Y(\alpha)Y(\beta)(X(\alpha) - X(\beta))}{X(\alpha)Y(\beta) - Y(\alpha)X(\beta)}$$

$$= \frac{-(Y(\alpha))^2 X'(\alpha)}{X(\alpha)Y'(\alpha) - Y(\alpha)X'(\alpha)}$$



We get the parametrization

$$\left(\frac{(X(\alpha))^2 Y'(\alpha)}{X(\alpha)Y'(\alpha) - Y(\alpha)X'(\alpha)}, \frac{-(Y(\alpha))^2 X'(\alpha)}{X(\alpha)Y'(\alpha) - Y(\alpha)X'(\alpha)} \right)$$

for the envelope curve.

Example:

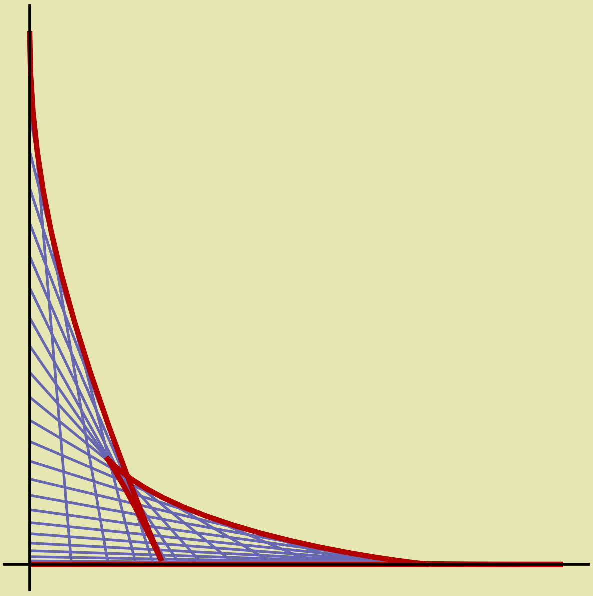
The picture shows lines generated by

$$X(\alpha) = 4\left(\alpha - \frac{1}{2}\right)^3 + \frac{1}{2}$$

along the x-axis and

$$Y(\alpha) = 1 - \alpha^2$$

along the y-axis.



The formula from the previous slide gives the parametrization

$$\left(-\frac{2\alpha^3(4\alpha^2 - 6\alpha + 3)}{4\alpha^4 - 15\alpha^2 + 12\alpha - 3}, -\frac{3(2\alpha - 1)^2(\alpha^2 - 1)^2}{4\alpha^4 - 15\alpha^2 + 12\alpha - 3}\right)$$

for the envelope curve.

Example:

A ladder of length L slides down a wall.

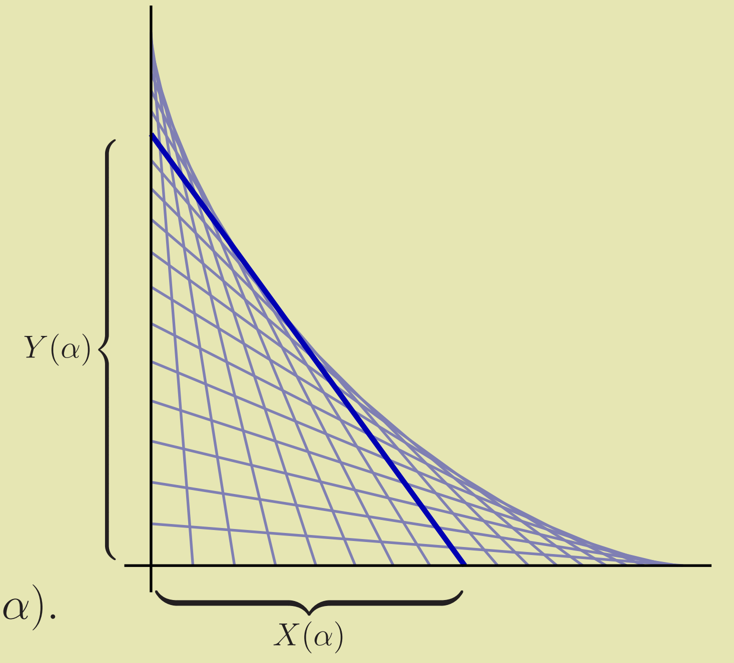
What is the envelope curve?

Solution:

We want $(X(\alpha))^2 + (Y(\alpha))^2 = L^2$,

so we may as well take

$X(\alpha) = L\sin(\alpha)$ and $Y(\alpha) = L\cos(\alpha)$.



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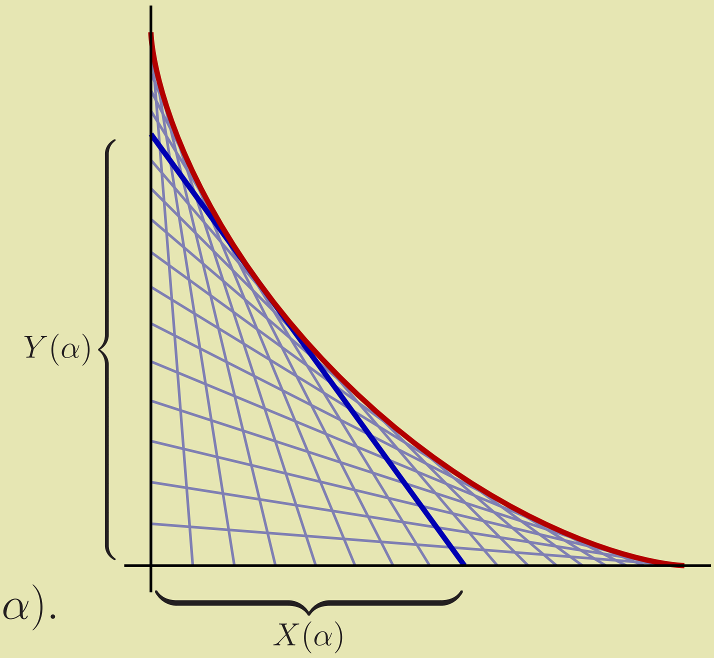


Solution:

We want $(X(\alpha))^2 + (Y(\alpha))^2 = L^2$,

so we may as well take

$X(\alpha) = L\sin(\alpha)$ and $Y(\alpha) = L\cos(\alpha)$.

We get

$$\left(\frac{(X(\alpha))^2 Y'(\alpha)}{X(\alpha)Y'(\alpha) - Y(\alpha)X'(\alpha)}, \frac{-(Y(\alpha))^2 X'(\alpha)}{X(\alpha)Y'(\alpha) - Y(\alpha)X'(\alpha)}\right)$$

$$= (L\sin^3(\alpha), L\cos^3(\alpha))$$

Remarks:

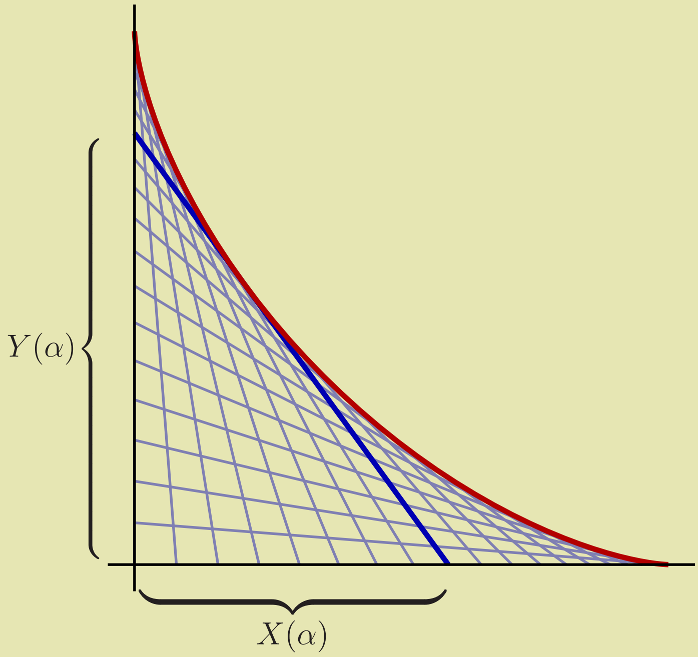
The envelope curve, parametrized by

$$x = L\sin^3(\alpha) \;\text{ and }\; y = L\cos^3(\alpha)$$

has equation

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = L^{\frac{2}{3}}$$

(This is called an *astroid*.)



Remarks:

The envelope curve, parametrized by

$$x = L\sin^3(\alpha) \quad \text{and} \quad y = L\cos^3(\alpha)$$

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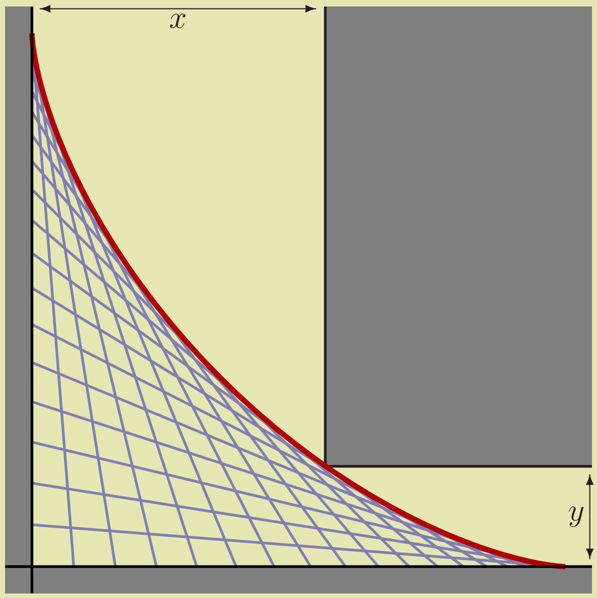
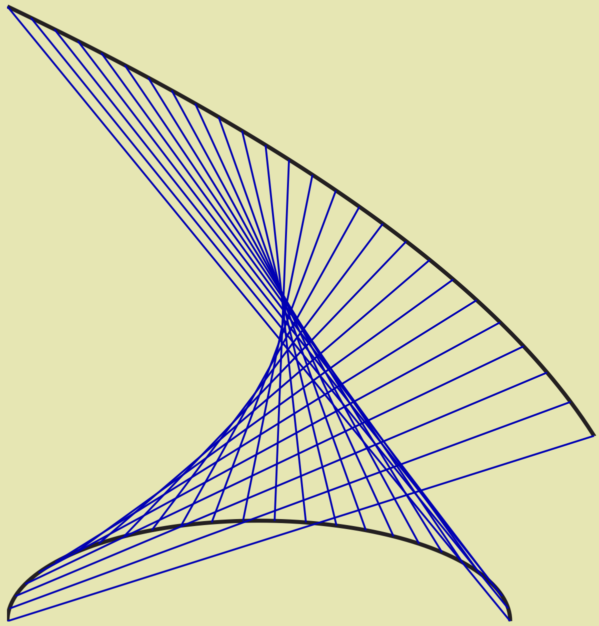


So if you want to carry your ladder around a corner from a hallway of width x into a hallway of width y, the length of the ladder has to satisfy

$$L^{\frac{2}{3}} \leq x^{\frac{2}{3}} + y^{\frac{2}{3}}$$

Further Generalization:

Instead of using the axes as nailing lines, use parametrized curves



$$(X_1(\alpha), Y_1(\alpha)) \text{ and } (X_2(\alpha), Y_2(\alpha))$$

Exercise:

Find the intersection point of ℓ_α and ℓ_β, and show that as $\beta \to \alpha$, this point approaches

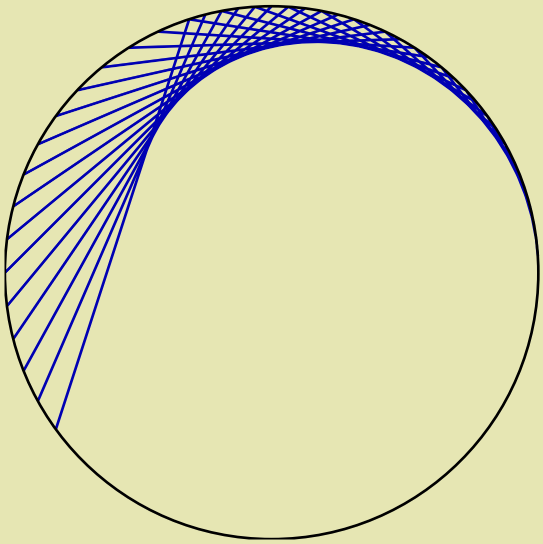
$$x = \frac{(X_1 X_2' - X_1' X_2)(Y_2 - Y_1) - (X_1 Y_2' - Y_1' X_2)(X_2 - X_1)}{(X_2' - X_1')(Y_2 - Y_1) - (Y_2' - Y_1')(X_2 - X_1)}$$

$$y = \frac{(Y_1 X_2' - X_1' Y_2)(Y_2 - Y_1) - (Y_1 Y_2' - Y_1' Y_2)(X_2 - X_1)}{(X_2' - X_1')(Y_2 - Y_1) - (Y_2' - Y_1')(X_2 - X_1)}$$

Example:

Let

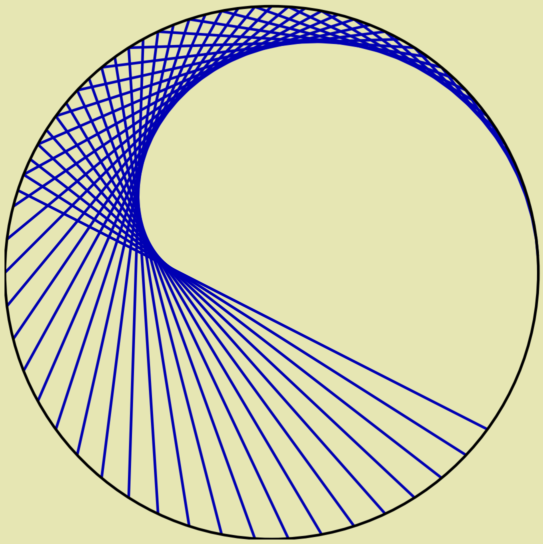
$$\begin{aligned} X_1(\alpha) &= \cos(\alpha) \\ Y_1(\alpha) &= \sin(\alpha) \\ X_2(\alpha) &= \cos(2\alpha) \\ Y_2(\alpha) &= \sin(2\alpha) \end{aligned}$$



Interpretations:

- Drive nails around a circle at regular intervals. Connect nail 1 to nail 2, 2 to 4, 3 to 6, 4 to 8, 5 to 10, and so on.
- (Simoson, 2000) Two runners set off around a circular track with a bungee cord stretched between them. The second runner goes twice as fast as the first.

Yet Another Exercise:



Substitute

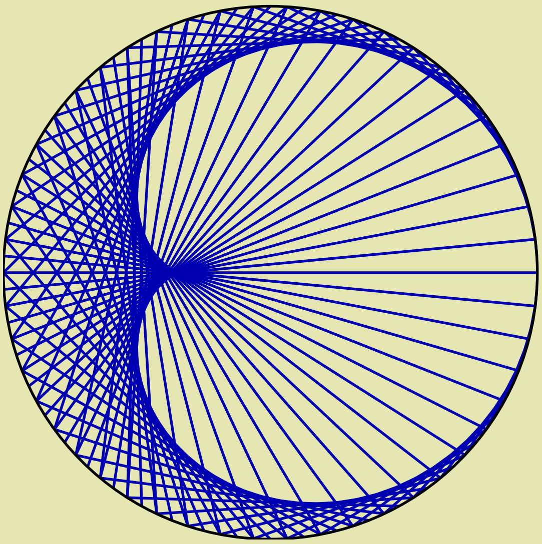
$$
\begin{aligned}
X_1(\alpha) &= \cos(\alpha)\\
Y_1(\alpha) &= \sin(\alpha)\\
X_2(\alpha) &= \cos(2\alpha)\\
Y_2(\alpha) &= \sin(2\alpha)
\end{aligned}
$$

into

$$
\begin{aligned}
x &= \frac{(X_1X_2' - X_1'X_2)(Y_2 - Y_1) - (X_1Y_2' - Y_1'X_2)(X_2 - X_1)}{(X_2' - X_1')(Y_2 - Y_1) - (Y_2' - Y_1')(X_2 - X_1)}\\
y &= \frac{(Y_1X_2' - X_1'Y_2)(Y_2 - Y_1) - (Y_1Y_2' - Y_1'Y_2)(X_2 - X_1)}{(X_2' - X_1')(Y_2 - Y_1) - (Y_2' - Y_1')(X_2 - X_1)}
\end{aligned}
$$

and simplify.

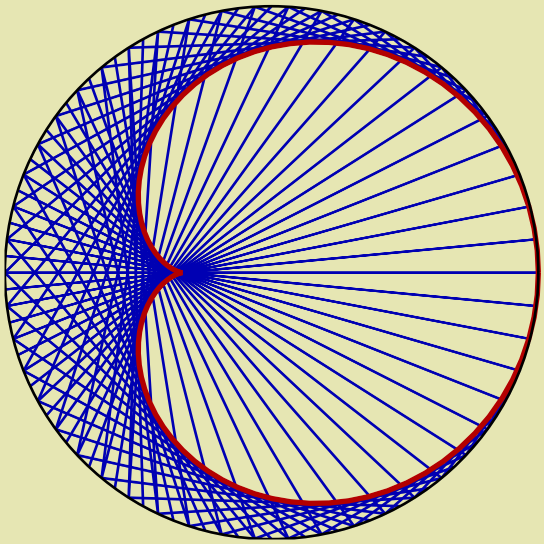
Answer:



$$x = \frac{\cos 2\alpha + 2\cos\alpha}{3}$$

$$y = \frac{\sin 2\alpha + 2\sin\alpha}{3}$$

Answer:



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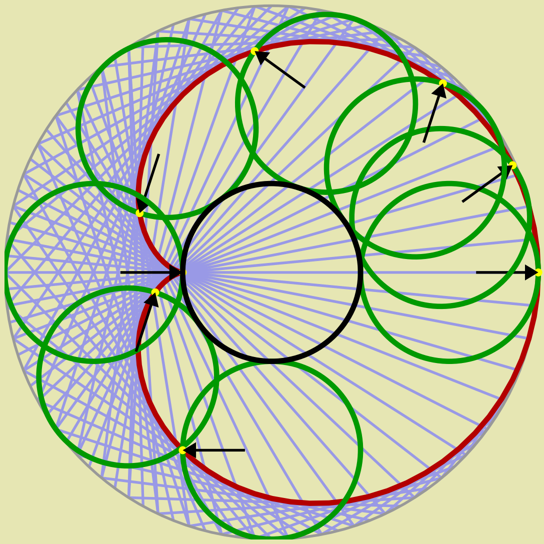


$$x = \frac{\cos 2\alpha + 2\cos\alpha}{3}$$

$$y = \frac{\sin 2\alpha + 2\sin\alpha}{3}$$

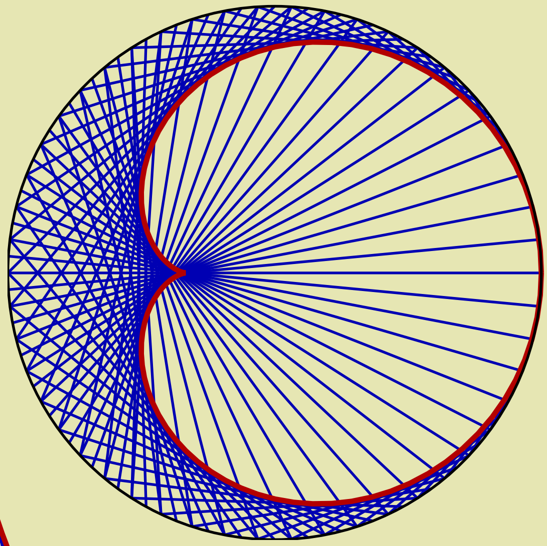
Write this as

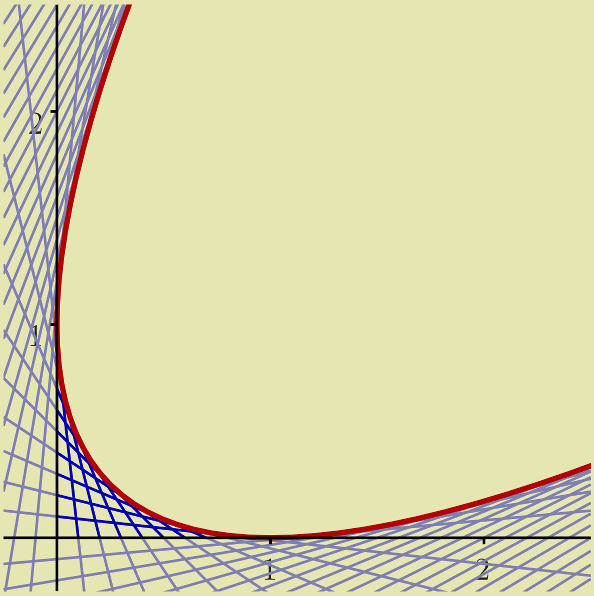
$$x = \frac{2}{3}\cos\alpha + \frac{1}{3}\cos 2\alpha, \quad y = \frac{2}{3}\sin\alpha + \frac{1}{3}\sin 2\alpha$$

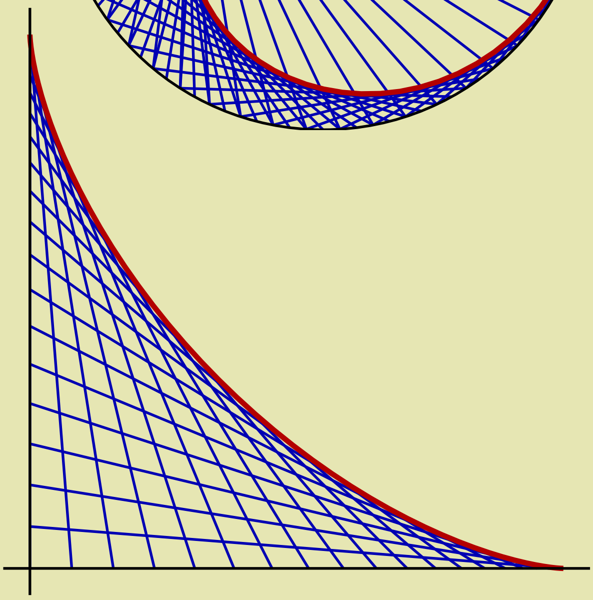
to see that our curve is an epicycloid, traced by a point on a circle of radius $\frac{1}{3}$ rolling around the outside of a fixed circle of radius $\frac{1}{3}$.

Conclusion:

The parabola, the astroid, and the epicycloid are all easy string-art curves.
Some other easy ones are the hyperbola and the circle.







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- GQ, Envelopes and String Art, to appear in *Mathematics Magazine*.
- Andrew J. Simoson, The trochoid as a tack in a bungee cord, *Mathematics Magazine* 73(3), 2000.
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