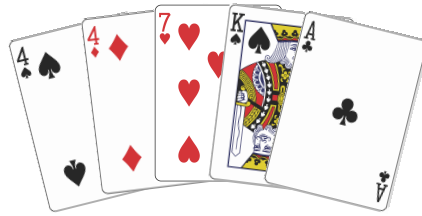
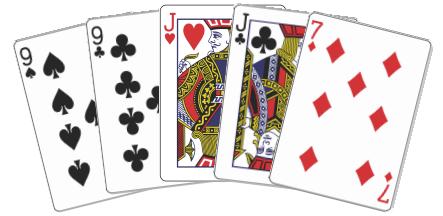


A Poker Paradox

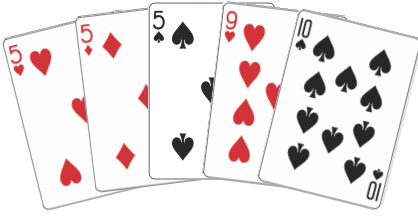
The nine types
of poker hands



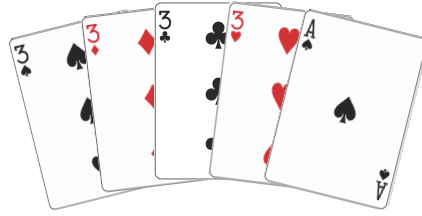
ONE PAIR



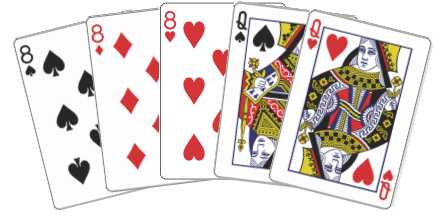
TWO PAIR



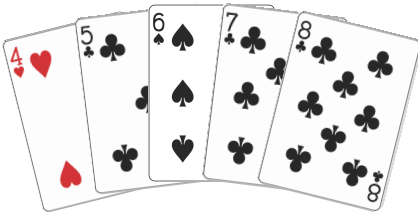
THREE OF A KIND



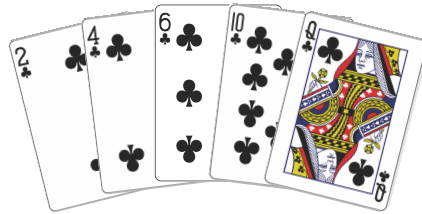
FOUR OF A KIND



FULL HOUSE



STRAIGHT



FLUSH



STRAIGHT FLUSH

and...



NOTHING

There are

$$\binom{52}{5} = 2\,598\,960$$

equally-likely poker hands.

Type of hand	Frequency
STRAIGHT FLUSH	40
FOUR OF A KIND	624
FULL HOUSE	3 744
FLUSH	5 108
STRAIGHT	10 200
THREE OF A KIND	54 912
TWO PAIR	123 552
ONE PAIR	1 098 240
NOTHING	1 302 540

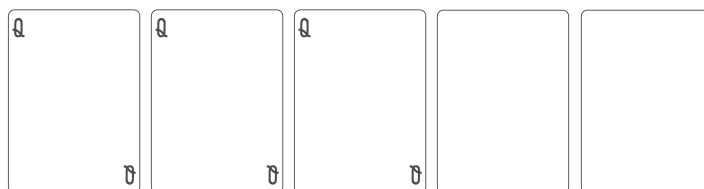
We rank them according to their frequencies.

$$P(\text{one pair}) \approx 0.423; \quad P(\text{straight flush}) \approx 1.54 \times 10^{-5}.$$

Counting: THREE OF A KIND

--	--	--	--	--

Counting: THREE OF A KIND



- Choose a rank for the triple

13

Counting: THREE OF A KIND



- Choose a rank for the triple

13

- Choose suits for the triple

$$\binom{4}{3}$$

Counting: THREE OF A KIND



- Choose a rank for the triple

$$13$$

- Choose suits for the triple

$$\binom{4}{3}$$

- Choose ranks for the other two cards

$$\binom{12}{2}$$

Counting: THREE OF A KIND



- Choose a rank for the triple

$$13$$

- Choose suits for the triple

$$\binom{4}{3}$$

- Choose ranks for the other two cards

$$\binom{12}{2}$$

- Choose suits for the other two cards

$$4^2$$

Counting: THREE OF A KIND



- Choose a rank for the triple

13

- Choose suits for the triple

$\binom{4}{3}$

- Choose ranks for the other two cards

$\binom{12}{2}$

- Choose suits for the other two cards

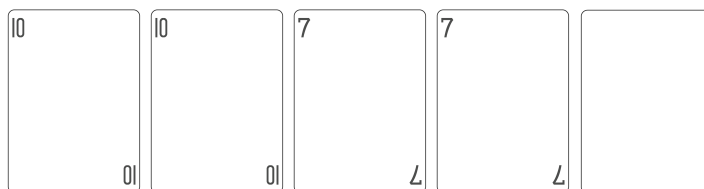
4^2

$$13 \times \binom{4}{3} \times \binom{12}{2} \times 4^2 = 54\,912$$

Counting: TWO PAIR



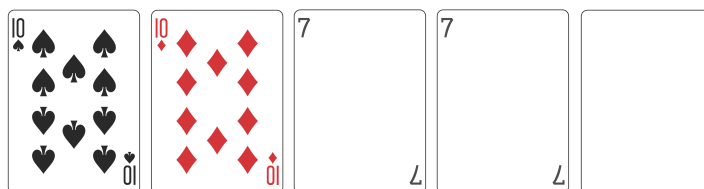
Counting: TWO PAIR



- Choose ranks for the pairs

$$\binom{13}{2}$$

Counting: TWO PAIR



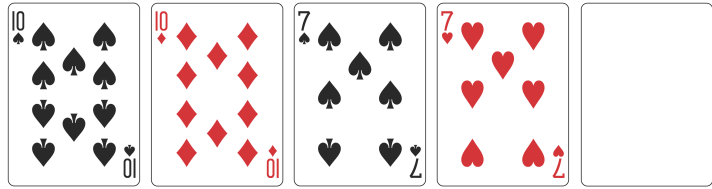
- Choose ranks for the pairs

$$\binom{13}{2}$$

- Choose suits for the higher pair

$$\binom{4}{2}$$

Counting: TWO PAIR



- Choose ranks for the pairs

$$\binom{13}{2}$$

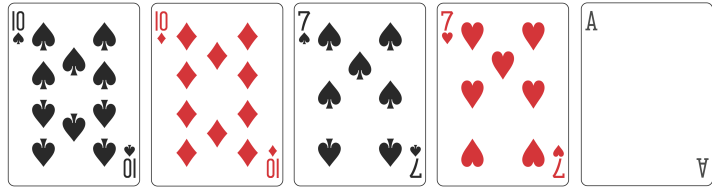
- Choose suits for the higher pair

$$\binom{4}{2}$$

- Choose suits for the lower pair

$$\binom{4}{2}$$

Counting: TWO PAIR



- Choose ranks for the pairs

$$\binom{13}{2}$$

- Choose suits for the higher pair

$$\binom{4}{2}$$

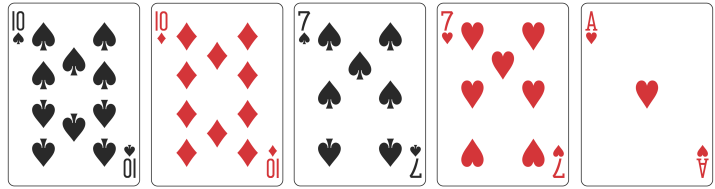
- Choose suits for the lower pair

$$\binom{4}{2}$$

- Choose a rank for the fifth card

$$11$$

Counting: TWO PAIR



- Choose ranks for the pairs

$$\binom{13}{2}$$

- Choose suits for the higher pair

$$\binom{4}{2}$$

- Choose suits for the lower pair

$$\binom{4}{2}$$

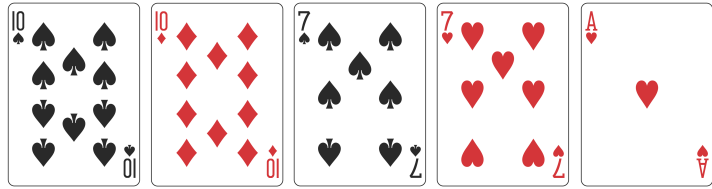
- Choose a rank for the fifth card

$$11$$

- Choose a suit for the fifth card

$$4$$

Counting: TWO PAIR



- Choose ranks for the pairs

$$\binom{13}{2}$$

- Choose suits for the higher pair

$$\binom{4}{2}$$

- Choose suits for the lower pair

$$\binom{4}{2}$$

- Choose a rank for the fifth card

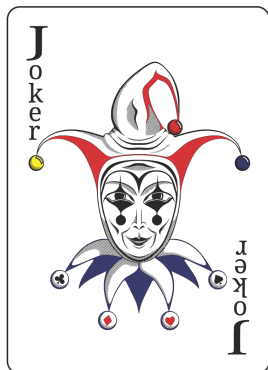
$$11$$

- Choose a suit for the fifth card

$$4$$

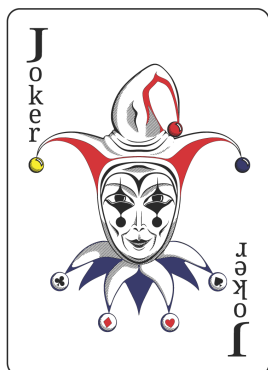
$$\binom{13}{2} \times \binom{4}{2} \times \binom{4}{2} \times 11 \times 4 = 123\,552$$

With one wild card, there are an additional $\binom{52}{4} = 270\,725$ hands.



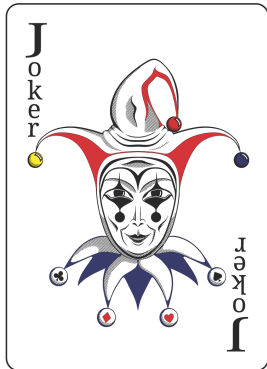
Type of hand	Frequency	
	no w.c.	one w.c.
FIVE OF A KIND	0	
STRAIGHT FLUSH	40	
FOUR OF A KIND	624	
FULL HOUSE	3 744	
FLUSH	5 108	
STRAIGHT	10 200	
THREE OF A KIND	54 912	
TWO PAIR	123 552	
ONE PAIR	1 098 240	
NOTHING	1 302 540	

With one wild card, there are an additional $\binom{52}{4} = 270\,725$ hands.



Type of hand	Frequency	
	no w.c.	one w.c.
FIVE OF A KIND	0	13
STRAIGHT FLUSH	40	
FOUR OF A KIND	624	
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STRAIGHT	10 200	
THREE OF A KIND	54 912	
TWO PAIR	123 552	
ONE PAIR	1 098 240	
NOTHING	1 302 540	0

With one wild card, there are an additional $\binom{52}{4} = 270\,725$ hands.



Type of hand	Frequency	
	no w.c.	one w.c.
FIVE OF A KIND	0	13
STRAIGHT FLUSH	40	164
FOUR OF A KIND	624	2 496
FULL HOUSE	3 744	2 808
FLUSH	5 108	2 696
STRAIGHT	10 200	10 332
THREE OF A KIND	54 912	82 368
TWO PAIR	123 552	0
ONE PAIR	1 098 240	169 848
NOTHING	1 302 540	0

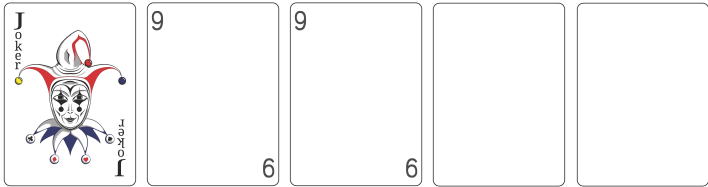
Counting: THREE OF A KIND

with one wild card



Counting: THREE OF A KIND

with one wild card

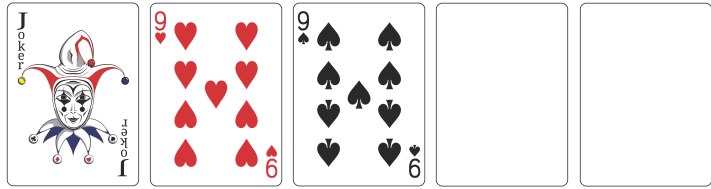


- Choose a rank for the pair

13

Counting: THREE OF A KIND

with one wild card



- Choose a rank for the pair

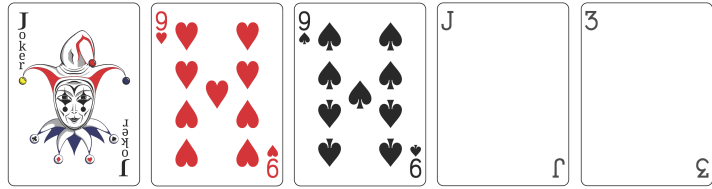
13

- Choose suits for the pair

$$\binom{4}{2}$$

Counting: THREE OF A KIND

with one wild card



- Choose a rank for the pair

$$13$$

- Choose suits for the pair

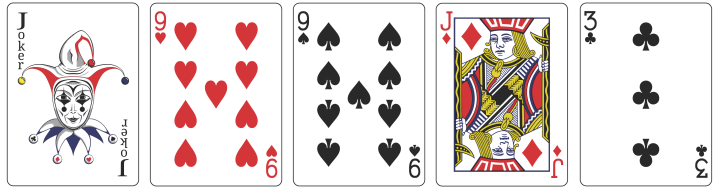
$$\binom{4}{2}$$

- Choose ranks for the other cards

$$\binom{12}{2}$$

Counting: THREE OF A KIND

with one wild card



- Choose a rank for the pair

$$13$$

- Choose suits for the pair

$$\binom{4}{2}$$

- Choose ranks for the other cards

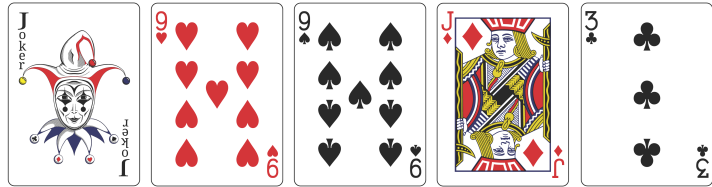
$$\binom{12}{2}$$

- Choose a suits for the other cards

$$4^2$$

Counting: THREE OF A KIND

with one wild card



- Choose a rank for the pair

13

- Choose suits for the pair

$\binom{4}{2}$

- Choose ranks for the other cards

$\binom{12}{2}$

- Choose a suits for the other cards

4^2

$$13 \times \binom{4}{2} \times \binom{12}{2} \times 4^2 = 82\,368.$$

Counting: TWO PAIR

with one wild card



Counting: TWO PAIR

with one wild card

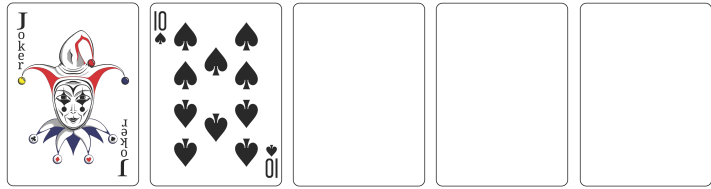


- Choose a rank to pair with the wild card

13

Counting: TWO PAIR

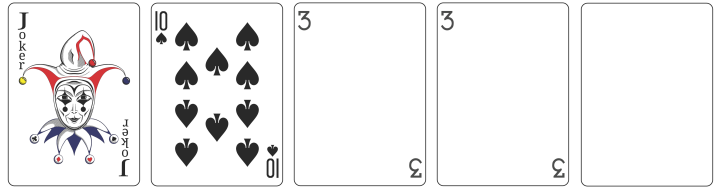
with one wild card



- Choose a rank to pair with the wild card 13
 - Choose a suit for the first pair card 4
-

Counting: TWO PAIR

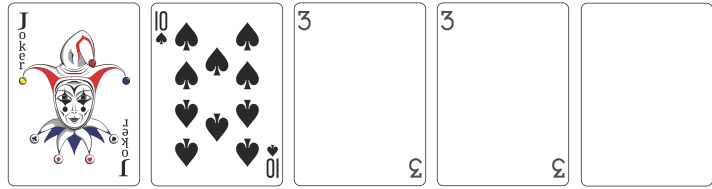
with one wild card



- Choose a rank to pair with the wild card 13
 - Choose a suit for the first pair card 4
 - Choose a rank for the other pair 12
-

Counting: TWO PAIR

with one wild card



- Choose a rank to pair with the wild card 13
 - Choose a suit for the first pair card 4
 - Choose a rank for the other pair 12
 - **STOP!** This hand is THREE OF A KIND. 0
-

Frequencies of five-card poker hands

Type of hand	no w.c.	one w.c.	Total
FIVE OF A KIND	0	13	13
STRAIGHT FLUSH	40	164	204
FOUR OF A KIND	624	2 496	3 120
FULL HOUSE	3 744	2 808	6 552
FLUSH	5 108	2 696	7 804
STRAIGHT	10 200	10 332	20 532
THREE OF A KIND	54 912	82 368	137 280
TWO PAIR	123 552	0	123 552
ONE PAIR	1 098 240	169 848	1 268 088
NOTHING	1 302 540	0	1 302 540

Frequencies of five-card poker hands

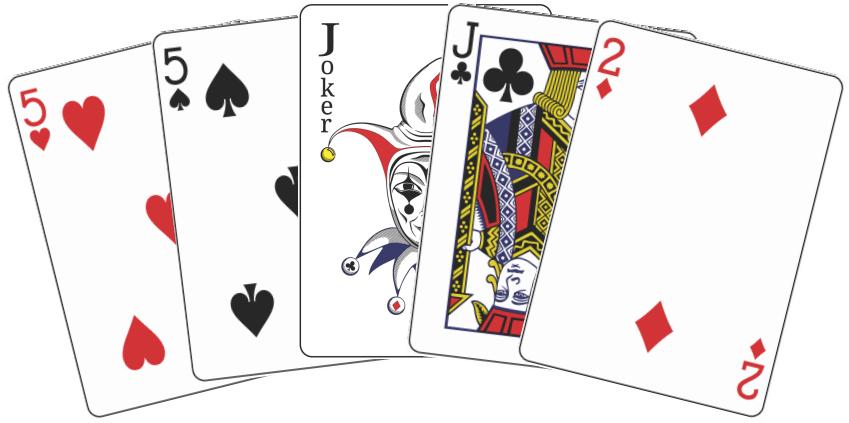
Type of hand	no w.c.	one w.c.	Total
FIVE OF A KIND	0	13	13
STRAIGHT FLUSH	40	164	204
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TWO PAIR	123 552	0	123 552
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NOTHING	1 302 540	0	1 302 540

Modified ranking of five-card poker hands

Type of hand	no w.c.	one w.c.	Total
FIVE OF A KIND	0	13	13
STRAIGHT FLUSH	40	164	204
FOUR OF A KIND	624	2 496	3 120
FULL HOUSE	3 744	2 808	6 552
FLUSH	5 108	2 696	7 804
STRAIGHT	10 200	10 332	20 532
TWO PAIR	123 552	82 368	??
THREE OF A KIND	54 912	82 368	??
ONE PAIR	1 098 240	169 848	1 268 088
NOTHING	1 302 540	0	1 302 540

The swing hands

There are 82 368
hands like this:



If TWO PAIR is ranked above THREE OF A KIND,
they count as **TWO PAIR**.

If THREE OF A KIND is ranked above TWO PAIR,
they count as **THREE OF A KIND**.

Modified ranking of five-card poker hands

Type of hand	no w.c.	one w.c.	Total
FIVE OF A KIND	0	13	13
STRAIGHT FLUSH	40	164	204
FOUR OF A KIND	624	2 496	3 120
FULL HOUSE	3 744	2 808	6 552
FLUSH	5 108	2 696	7 804
STRAIGHT	10 200	10 332	20 532
TWO PAIR	123 552	82 368	205 920
THREE OF A KIND	54 912	0	54 912
ONE PAIR	1 098 240	169 848	1 268 088
NOTHING	1 302 540	0	1 302 540

Modified ranking of five-card poker hands

Type of hand	no w.c.	one w.c.	Total
FIVE OF A KIND	0	13	13
STRAIGHT FLUSH	40	164	204
FOUR OF A KIND	624	2 496	3 120
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Modified ranking of five-card poker hands

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Modified ranking of five-card poker hands

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FIVE OF A KIND	0	13	13
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FULL HOUSE	3 744	2 808	6 552
FLUSH	5 108	2 696	7 804
STRAIGHT	10 200	10 332	20 532
TWO PAIR	123 552	82 368	205 920
THREE OF A KIND	54 912	0	54 912
ONE PAIR	1 098 240	169 848	1 268 088
NOTHING	1 302 540	0	1 302 540

The Paradox: With one wild card, there is no ranking of the types of poker hands that is consistent with their frequencies.

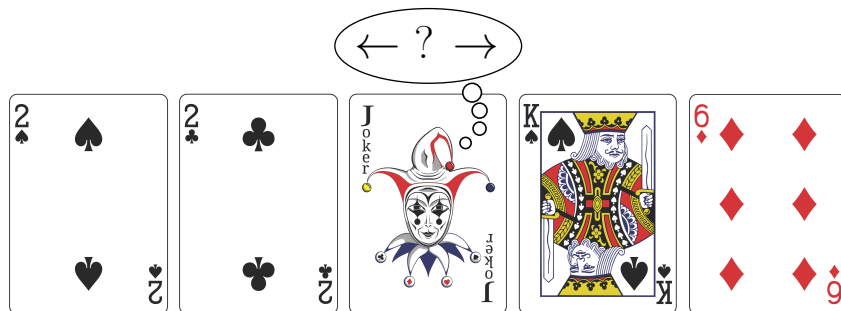
Either

TWO PAIR

or

THREE OF A KIND

will have both a higher ranking *and*
and a higher frequency than the other.



Frequencies of poker hands with two wild cards

Type of hand	no w.c.	w.c.	Total
FIVE OF A KIND	0	78	78
STRAIGHT FLUSH	40	584	624
FOUR OF A KIND	624	8 736	9 360
FULL HOUSE	3 744	5 616	9 360
FLUSH	5 108	6 280	11 388
STRAIGHT	10 200	24 504	34 704
THREE OF A KIND	54 912	178 056	232 968
TWO PAIR	123 552	0	123 552
ONE PAIR	1 098 240	339 696	1 437 936
NOTHING	1 302 540	0	1 302 540

Frequencies of poker hands with two wild cards

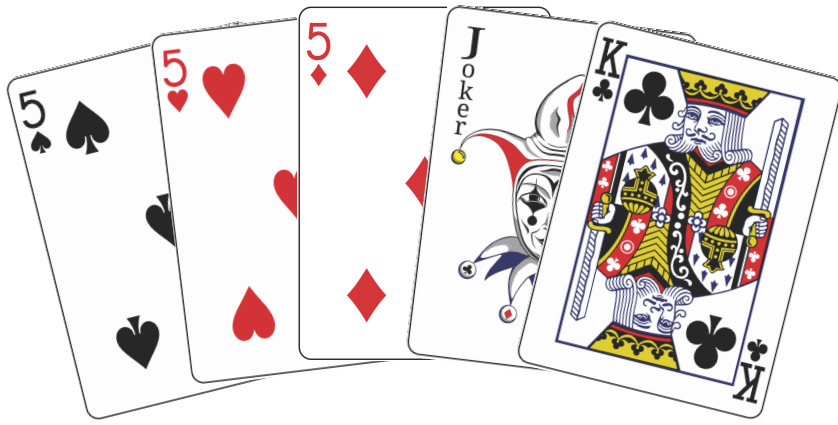
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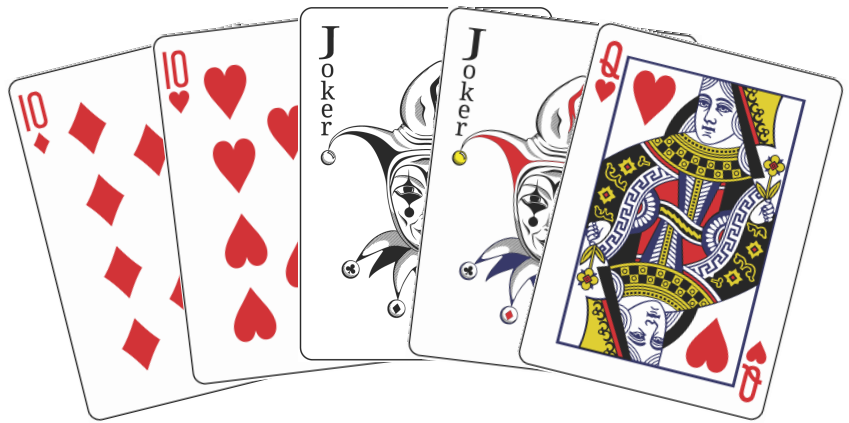
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FOUR OF A KIND	624	8 736	9 360
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FLUSH	5 108	6 280	11 388
STRAIGHT	10 200	24 504	34 704
THREE OF A KIND	54 912	178 056	232 968
TWO PAIR	123 552	0	123 552
ONE PAIR	1 098 240	339 696	1 437 936
NOTHING	1 302 540	0	1 302 540



FULL HOUSE
OR
FOUR OF A KIND?

There are 8 736 hands
willing to join
either camp.



Frequencies of poker hands with three wild cards

Type of hand	no w.c.	w.c.	Total
FIVE OF A KIND	0	273	273
STRAIGHT FLUSH	40	1444	1484
FOUR OF A KIND	624	19 784	20 408
FULL HOUSE	3 744	8 424	12 168
FLUSH	5 108	10 752	15 860
STRAIGHT	10 200	42 516	52 716
THREE OF A KIND	54 912	287 064	341 796
TWO PAIR	123 552	0	123 552
ONE PAIR	1 098 240	509 544	1 607 784
NOTHING	1 302 540	0	1 302 540



References

- Steve Gadbois, “Poker with Wild Cards – A Paradox?” *Mathematics Magazine* 69(4), October 1996, pp. 283-285.
- Albert H. Morehead and Geoffrey Mott-Smith, editors, *Hoyle Up-to-date: Official Rules for All Important Games*, Grosset & Dunlap, New York, 1970.