Machin's Trick

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Theorem:

If $-1 \leq x \leq 1$, then

$$\tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots$$

James Gregory (1671)

Gottfried Leibniz (1673)

Madhāvi of Sangamagramma, Kerala (c. 1370)

Application:

Since $\tan^{-1}(1) = \frac{\pi}{4}$, we have

$$\pi = 4\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots\right]$$

Theorem (Alternating Series Test):

If $a_1 > a_2 > \cdots > a_k > \cdots > 0$ and $\lim_{k\to\infty} a_k = 0$,

then $\sum_{k=1}^{\infty} (-1)^{k+1} a_k$ converges to a real number S and

$$\left| S - \sum_{k=1}^{n} (-1)^{k+1} a_k \right| < a_{n+1}.$$

AST can be good news ...

We know $e^{-1} = \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \cdots$, so

e^{-1} differs from, say, $\frac{1}{2!} - \frac{1}{3!} + \cdots - \frac{1}{9!}$ by less than $\frac{1}{10!} \approx 2.8 \times 10^{-7}$.

... or bad news:

Using $\pi = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \frac{4}{9} - \cdots$, in order to guarantee an error of less than 5×10^{-11} (for ten correct decimal places), we would need

$$\frac{4}{2n+1} < 5 \times 10^{-11}.$$

We need to sum 40,000,000,000 terms.

Hard work

Computers in the seventeenth century were not very fast, and required lots of ink and quills, or slate and chalk.

$\pi = 4$	4.0000000000
$-4/3$	-1.3333333333
	2.6666666667
$+4/5$	$+0.8000000000$
	3.4666666667
$-4/7$	-0.5714285714
	2.8952380953
$+4/9$	$+0.4444444444$
	3.3396825397
$-4/11$	-0.3636363636
	2.9760461761
$+4/13$	$+0.3076923077$
	3.2837384838

Less work

The series

$$\tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots$$

converges more rapidly when x is small.

Using $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$, Edmund Halley developed the series (earlier known to Madhāvi)

$$\pi = 6\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = 2\sqrt{3}\left(1 - \frac{1}{3}\cdot\frac{1}{3} + \frac{1}{5}\cdot\frac{1}{3^2} - \frac{1}{7}\cdot\frac{1}{3^3} + \frac{1}{9}\cdot\frac{1}{3^4} - \cdots\right)$$

The Halley-Madhāvi series gives ten-digit precision with only 20 terms. (But you have to know $\sqrt{3}$ to at least as many places.)

The lion

Newton observed that

$$\pi = 4\tan^{-1}\left(\frac{1}{2}\right) + 2\tan^{-1}\left(\frac{4}{7}\right) + 2\tan^{-1}\left(\frac{1}{8}\right).$$

Evaluating each of these arctangents using the Gregory-Leibniz-Madhāvi series, we get

$$\begin{aligned}\pi \;=\; & \left(\frac{4}{2} + \frac{2\cdot 4}{7} + \frac{2}{8}\right) - \left(\frac{4}{3\cdot 2^3} + \frac{2\cdot 4^3}{3\cdot 7^3} + \frac{2}{3\cdot 8^3}\right) \\ & + \left(\frac{4}{5\cdot 2^5} + \frac{2\cdot 4^5}{5\cdot 7^5} + \frac{2}{5\cdot 8^5}\right) - \left(\frac{4}{7\cdot 2^7} + \frac{2\cdot 4^7}{7\cdot 7^7} + \frac{2}{7\cdot 8^7}\right) + \cdots\end{aligned}$$

The Newton series gives ten-digit precision with only 19 terms.
(But each term is a sum of three numbers.)

Sums of arctangents

$\pi =$	Discoverer	Terms for 10^{-100}
$4\tan^{-1}\frac{1}{2} + 2\tan^{-1}\frac{4}{7} + 2\tan^{-1}\frac{1}{8}$	Newton (1676)	202
$8\tan^{-1}\frac{1}{2} - 4\tan^{-1}\frac{1}{7}$	Hermann (1706)	164
$8\tan^{-1}\frac{1}{3} + 4\tan^{-1}\frac{1}{7}$	Hutton (1776)	104
$8\tan^{-1}\frac{1}{5} + 4\tan^{-1}\frac{1}{7} + 2\tan^{-1}\frac{1}{8}$	von Vega (1789)	71
$16\tan^{-1}\frac{1}{5} - 4\tan^{-1}\frac{1}{239}$	Machin (1706)	71
$48\tan^{-1}\frac{1}{18} + 32\tan^{-1}\frac{1}{57} - 5\tan^{-1}\frac{1}{239}$	Gauss	40
$32\tan^{-1}\frac{1}{10} - 4\tan^{-1}\frac{1}{239} + 16\tan^{-1}\frac{1}{515}$	Klingenstierna (1730)	50
$24\tan^{-1}\frac{1}{6} + 8\tan^{-1}\frac{1}{57} + 4\tan^{-1}\frac{1}{239}$	Størmer (1896)	64

Using Machin's series

In addition to its rapid convergence, Machin's series benefits tremendously from the fact that

$$\frac{1}{5} = \frac{2}{10}.$$

$\tan^{-1}(1/5) =$

$1/5 \times 10^{0}$	0.2000000000
$-8/3 \times 10^{-3}$	−0.0026666667
	0.1973333333
$+32/5 \times 10^{-5}$	+0.0000640000
	0.1973973333
$-128/7 \times 10^{-7}$	−0.0000018286
	0.1973955047
$+512/9 \times 10^{-9}$	+0.0000000569
	0.1973955616

$$\tan^{-1}\frac{1}{5} = \frac{2}{10} - \frac{1}{3}\cdot\frac{2^3}{10^3} + \frac{1}{5}\cdot\frac{2^5}{10^5} - \frac{1}{7}\cdot\frac{2^7}{10^7} + \frac{1}{9}\cdot\frac{2^9}{10^9} - \cdots$$

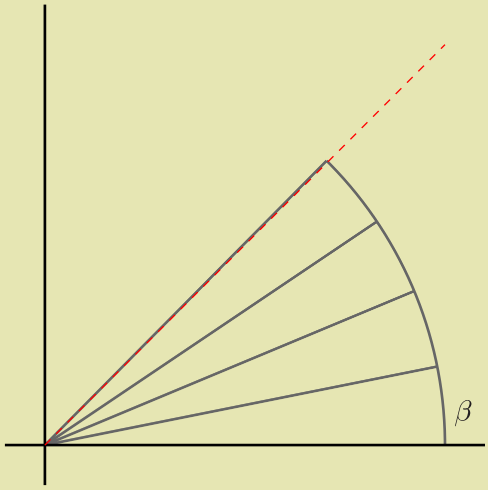
$$= \frac{2}{10} - \frac{8}{3}\cdot 10^{-3} + \frac{32}{5}\cdot 10^{-5} - \frac{128}{7}\cdot 10^{-7} + \frac{512}{9}\cdot 10^{-9} - \cdots$$

Machin's formula: the details

From

$$\tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$$

one easily derives



$$\tan 2\beta = \frac{2\tan\beta}{1-\tan^2\beta}$$

and

$$\tan 4\beta = \frac{4\tan\beta-4\tan^3\beta}{1-6\tan^2\beta+\tan^4\beta}$$

With $\beta = \tan^{-1}\frac{1}{5}$, we get $\tan 4\beta = \frac{4(1/5)-4(1/5)^3}{1-6(1/5)^2+(1/5)^4} = \frac{120}{119}$.

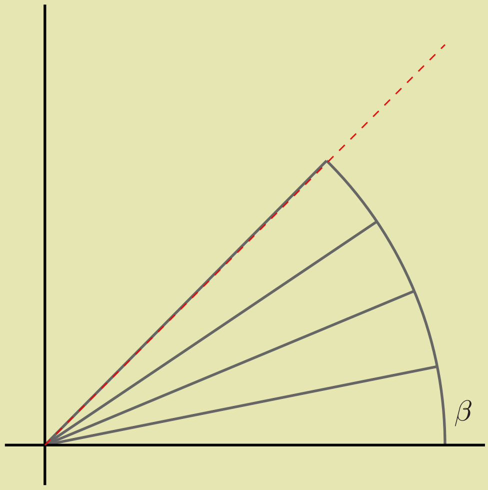
Machin's formula: the details

So

$$4\beta = \tan^{-1}\frac{120}{119} = \frac{\pi}{4} + \gamma$$

for some "make-up" angle γ. We get

$$\begin{aligned} \gamma &= 4\beta - \frac{\pi}{4} \\ \tan\gamma &= \frac{\tan 4\beta - \tan(\pi/4)}{1 + \tan 4\beta \tan(\pi/4)} \\ &= \frac{(120/119) - 1}{1 + (120/119)} = \frac{1}{119 + 120} \end{aligned}$$



Thus Machin's formula:

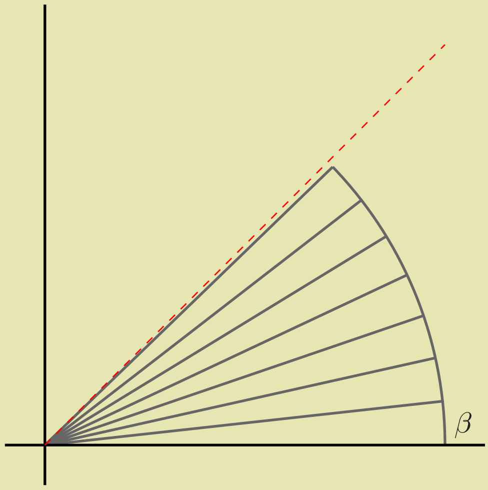
$$\frac{\pi}{4} = 4\beta - \gamma = 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239}$$

Can we do better?

Can we find an angle $\beta = \tan^{-1}\frac{1}{m}$ so that

$$k\beta \approx \frac{\pi}{4}$$

for some integer k?



... and for which the make-up angle γ has some simple expression, like $\tan^{-1}\frac{1}{p}$?

Generalization:

Using the addition formula

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta},$$

one derives (after some work)

$$\tan(n\beta) \;=\; \frac{\binom{n}{1} b - \binom{n}{3} b^3 + \binom{n}{5} b^5 - \binom{n}{7} b^7 + \cdots}{1 - \binom{n}{2} b^2 + \binom{n}{4} b^4 - \binom{n}{6} b^6 + \binom{n}{8} b^8 + \cdots}$$

where $b = \tan\beta$.

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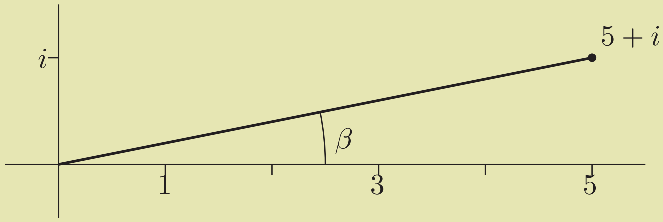
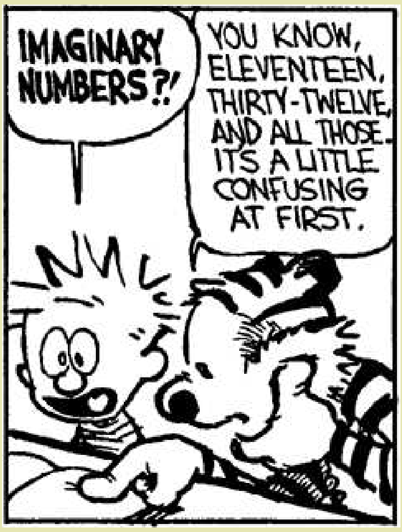
where $b=\tan\beta$.

Note the “broken binomial” with sign pattern $+,+,-,-,+,+,-,-$.
This suggests the calculation might be simpler working in …

The complex plane:

If $\beta = \tan^{-1}\dfrac{1}{5}$, then $\quad \beta = \arg(5+i)$.





We will also need the

Theorem: For complex numbers z_1 and z_2,

$$\arg(z_1) + \arg(z_2) = \arg(z_1 z_2) \quad \text{and} \quad \arg(z_1) - \arg(z_2) = \arg\frac{z_1}{z_2}$$

where the addition is taken modulo 2π.

The complex plane:

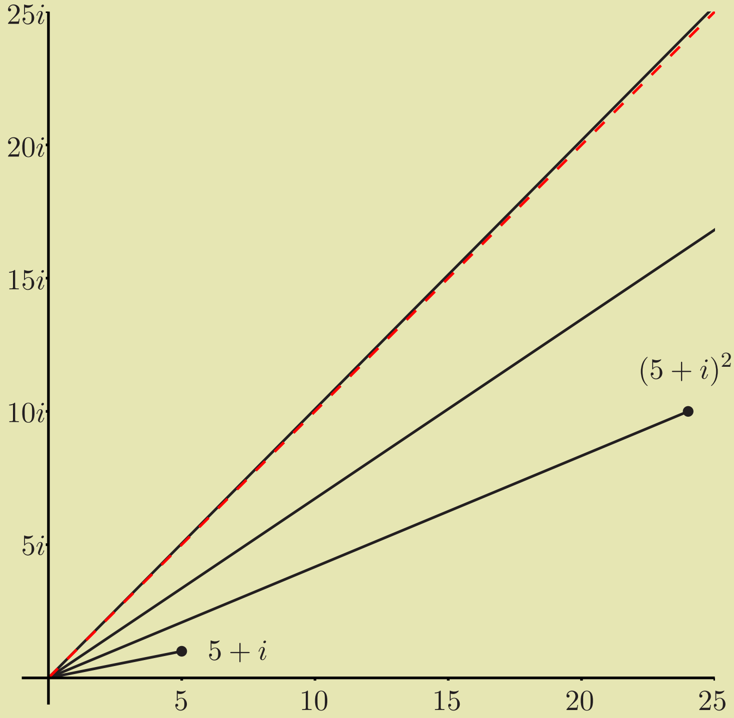
With $\beta = \arg(5+i)$, we get

$$\begin{aligned} 4\beta &= \arg((5+i)^4) \\ &= \arg(476+480i) \\ &= \arg(119+120i). \end{aligned}$$

So the "make-up" angle γ is $\arg(c)$, where

$$(1+i)c = 119+120i.$$

We get



$$c = \frac{119+120i}{1+i} = \frac{(119+120i)(1-i)}{(1+i)(1-i)} = \frac{239+i}{2}$$

so that $\gamma = \tan^{-1}\dfrac{1}{239}$.

Generalization:

You get a Machin-line formula

$$\frac{\pi}{4} = k_1 \tan^{-1}\frac{1}{m_1} + k_2 \tan^{-1}\frac{1}{m_2} + \cdots + k_r \tan^{-1}\frac{1}{m_r}$$

when the product

$$(m_1 + i)^{k_1}(m_2 + i)^{k_2}\cdots(m_r + i)^{k_r}$$

is an integer multiple of $(1 + i)$.

This makes a computer search easier. Some formulas I found:

$$\begin{aligned}\frac{\pi}{4} &= \tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{8} \\ &= 2\tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{7} + 2\tan^{-1}\frac{1}{8} \\ &= \tan^{-1}\frac{1}{5} + 2\tan^{-1}\frac{1}{7} + 2\tan^{-1}\frac{1}{8} + \tan^{-1}\frac{1}{18}\end{aligned}$$

Generalization:
The (integer) equation

$$(m_1+i)^{k_1}(m_2+i)^{k_2}\cdots(m_r+i)^{k_r}=N(1+i)$$

also means that one can apply number-theoretic techniques in $\mathbb{Z}[i]$ to the search for Machin-like formulas. One result is *Gravé's Problem*:

Question: What are the integer solutions to

$$k_1\tan^{-1}\frac{1}{m_1}+k_2\tan^{-1}\frac{1}{m_2}=N\frac{\pi}{4}?$$

Answer: (Størmer, 1885) There are only four primitive solutions (that is, with $\gcd(k_1,k_2)=1$). We've seen them all; Machin's is the most useful.

$$\frac{\pi}{4} = \tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3} = 2\tan^{-1}\frac{1}{2}-\tan^{-1}\frac{1}{7} = 2\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{7} = 4\tan^{-1}\frac{1}{5}-\tan^{-1}\frac{1}{239}$$

Project:

Look for Machin-line formulas

$$\frac{\pi}{4} = k_1 \tan^{-1}\frac{1}{m_1} + k_2 \tan^{-1}\frac{1}{m_2} + \cdots + k_r \tan^{-1}\frac{1}{m_r}$$

(or $(m_1+i)^{k_1}(m_2+i)^{k_2}\cdots(m_r+i)^{k_r} = N(1+i)$) with $r \geq 3$. Or prove **Størmer's Conjecture:** There exist only finitely many Machin-like formulas with $r = 3$.

Final comment: In 2002, Yasumasa Kanada computed a record number of decimal digits of π, surpassing the Borweins and the Chudnovskis, using the Machin-like formulas:

$$\begin{aligned}\pi &= 48\tan^{-1}\frac{1}{49} + 128\tan^{-1}\frac{1}{57} - 20\tan^{-1}\frac{1}{239} + 48\tan^{-1}\frac{1}{110443}\\ &= 176\tan^{-1}\frac{1}{157} + 28\tan^{-1}\frac{1}{239} - 48\tan^{-1}\frac{1}{682} + 96\tan^{-1}\frac{1}{12943}.\end{aligned}$$

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