Site-swap Juggling

Ingredients:

- Two hands (L and R)
- Some balls to throw
- A clock, ticking through the integers

$$\ldots, -2, -1, 0, 1, 2, \ldots$$

- A sequence $\ldots, h_{-2}, h_{-1}, h_0, h_1, h_2, \ldots$ of throw heights

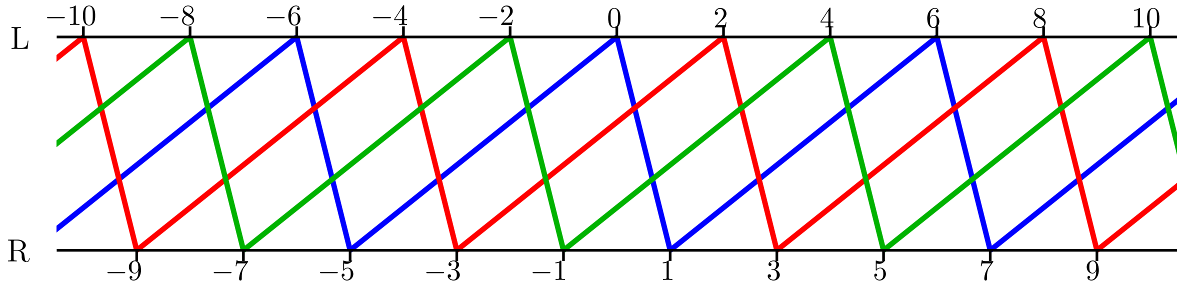
Process:

- The hands throw alternately, …, L, R, L, R, …, one throw per tick of the clock
- The ball thrown at time t is next thrown at time $t + h_t$, so it's in the air for (a little less than) h_t ticks.

Example:

Let $h_t = \begin{cases} 5 & \text{if } t \text{ is odd} \\ 1 & \text{if } t \text{ is even} \end{cases}$

$$\ldots, 5, 1, 5, 1, 5, 1, \ldots$$



Remarks:

- Since each hand can catch only one ball at a time, we can't have

$$t_1 + h_{t_1} = t_2 + h_{t_2} \quad \text{for} \quad t_1 \neq t_2$$

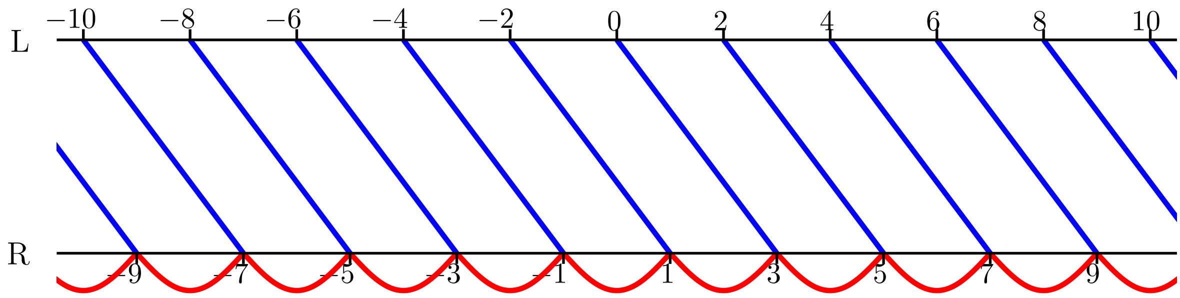
  That is, the map $t \mapsto t + h_t$ should be <u>injective</u>.

- At each tick of the clock, there should be a ball ready to throw.

  That is, the map $t \mapsto t + h_t$ should be <u>surjective</u>.

Non-example:

Let $h_t = \ldots, 3, 2, 3, 2, 3, 2, \ldots$



Definition: A sequence

$$\ldots, h_{-2}, h_{-1}, h_0, h_1, h_2, \ldots$$

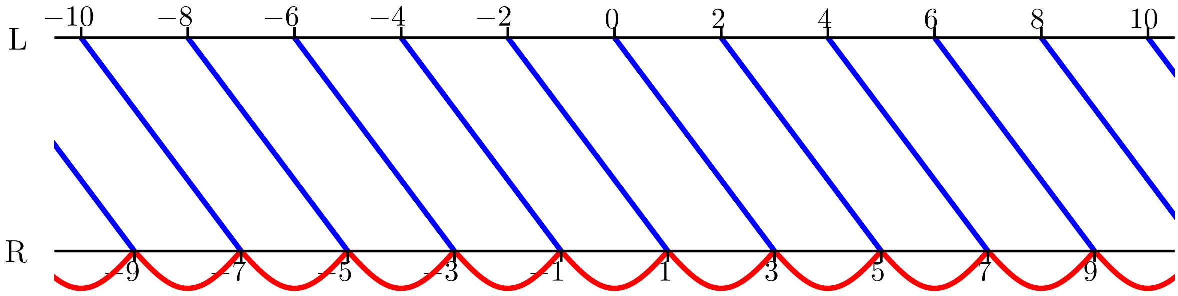
is a *juggling pattern* if the map

$$t \;\mapsto\; t + h_t$$

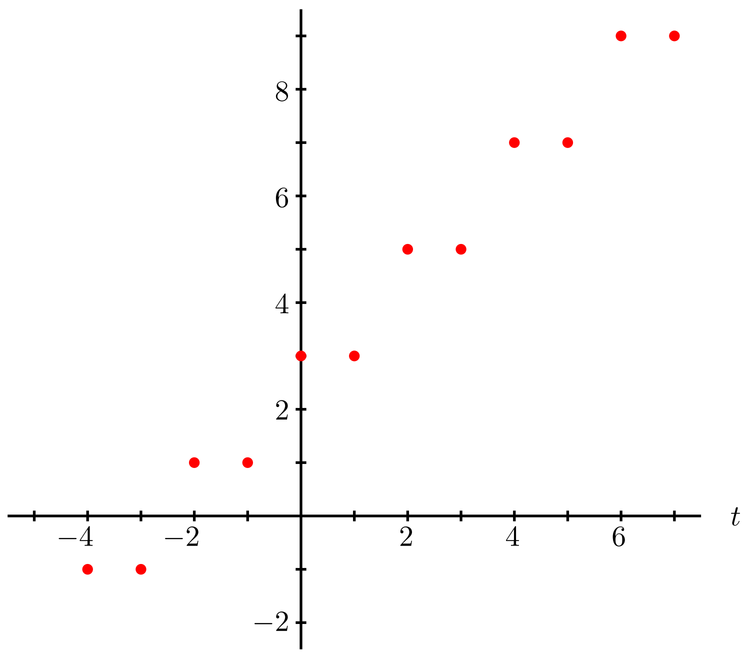
is a bijection from $\mathbb{Z}$ to $\mathbb{Z}$

Non-example: $h_t = \begin{cases} 3 & \text{if } t \text{ is even} \\ 2 & \text{if } t \text{ is odd} \end{cases}$

$$\ldots, 3, 2, 3, 2, 3, 2, \ldots$$



t	h_t	$t + h_t$
$\vdots$	$\vdots$	$\vdots$
-2	3	1
-1	2	1
0	3	3
1	2	3
2	3	5
3	2	5
$\vdots$	$\vdots$	$\vdots$

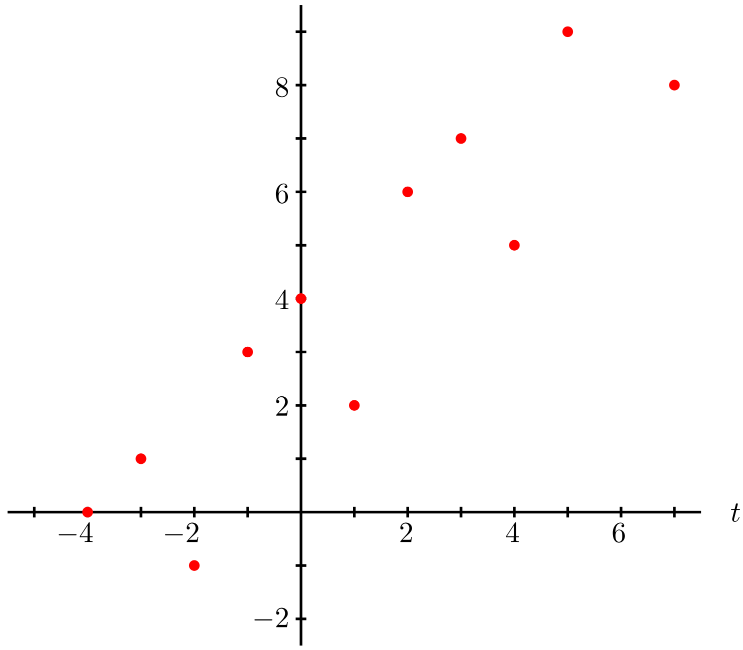


The map $t \mapsto t + h_t$ is clearly not bijective.

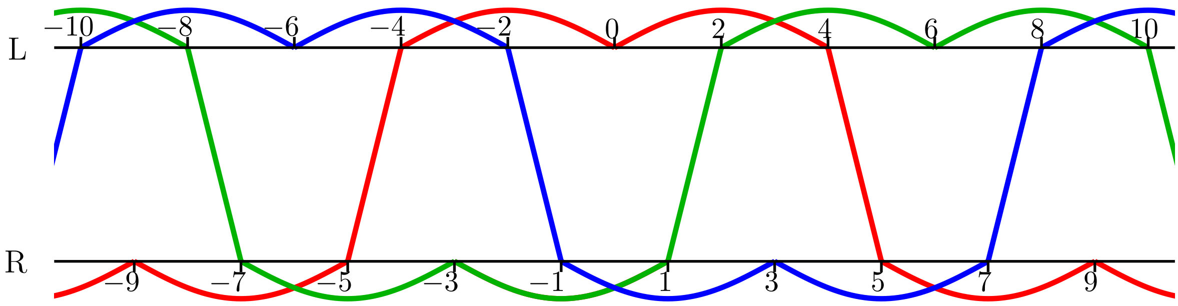
Example:

$$h_t = \ldots 4, 1, 4, 4, 1, 4, 4, 1, 4, 4, 1, 4, \ldots$$

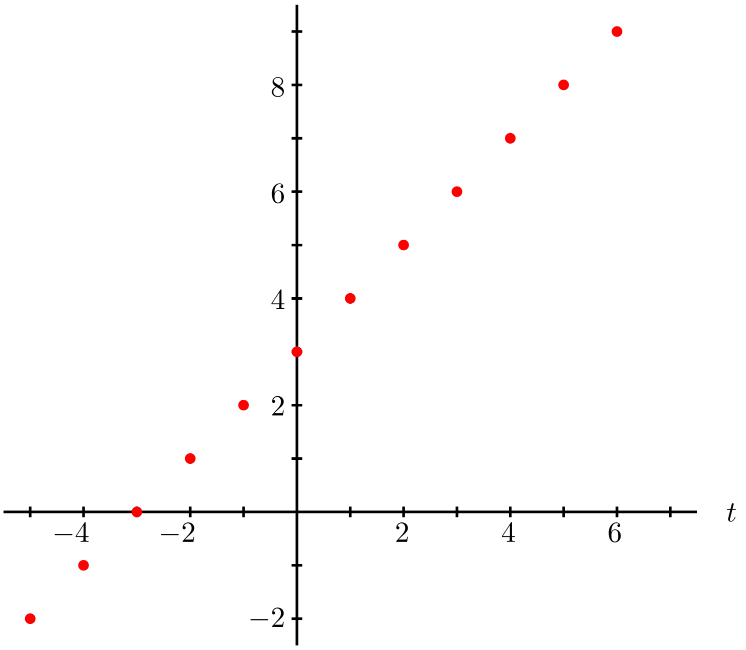
t	h_t	$t + h_t$
$\vdots$	$\vdots$	$\vdots$
-3	4	1
-2	1	-1
-1	4	3
0	4	4
1	1	2
2	4	6
3	4	7
4	1	5
5	4	9
$\vdots$	$\vdots$	$\vdots$



The map $t \mapsto t + h_t$ appears to be bijective.



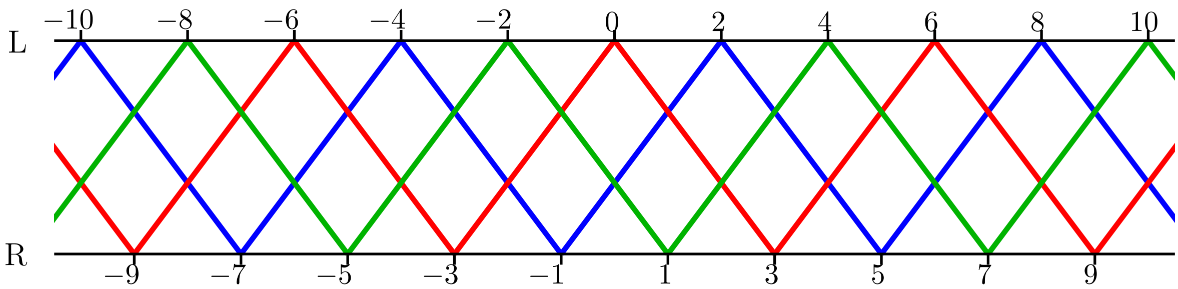
Example: $h_t = 3$ for all t



The map

$$t \;\mapsto\; t + h_t$$

is clearly bijective.

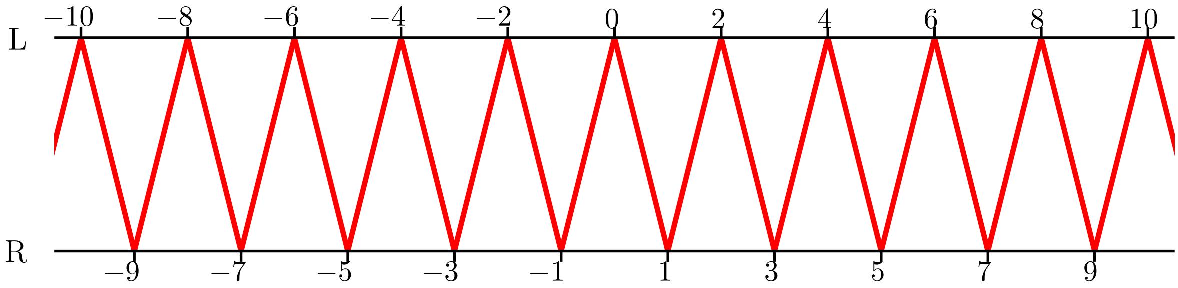


Family of examples:

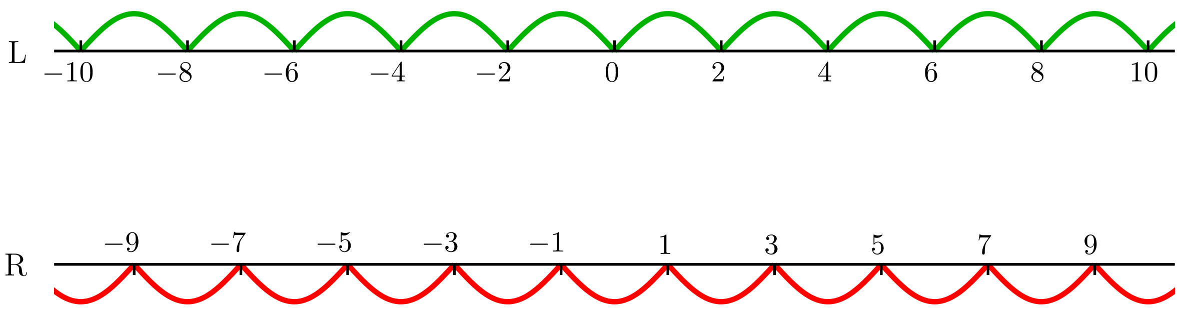
$$h_t \;=\; \ldots, c, c, c, c, c, \ldots$$

is always a juggling pattern for any positive integer c.

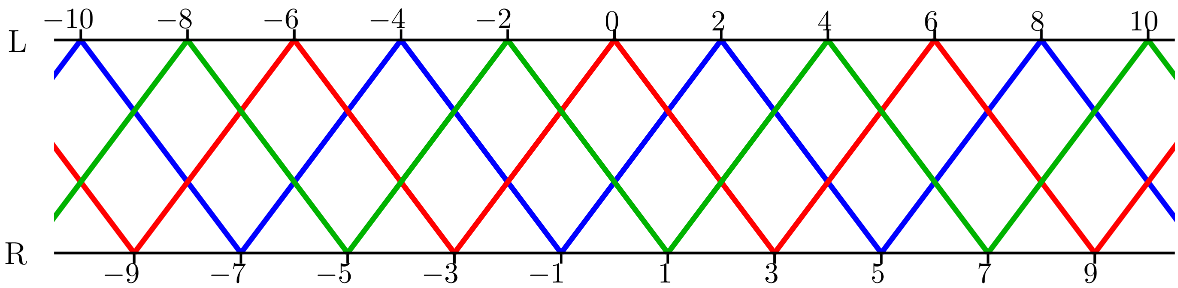
$c = 1$



$c = 2$

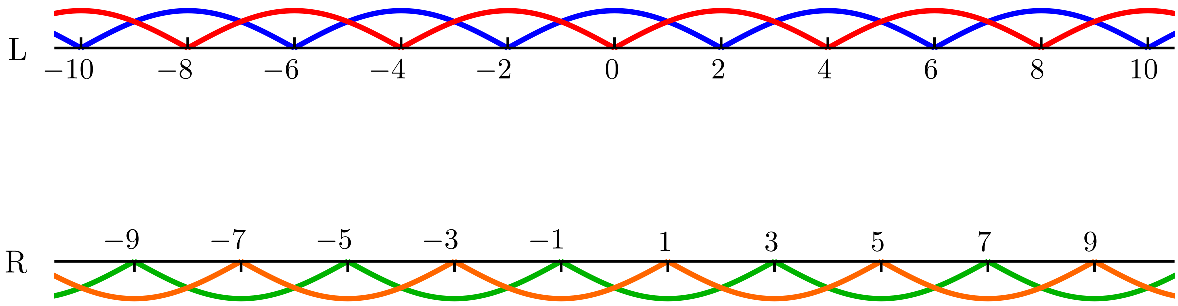


$c = 3$

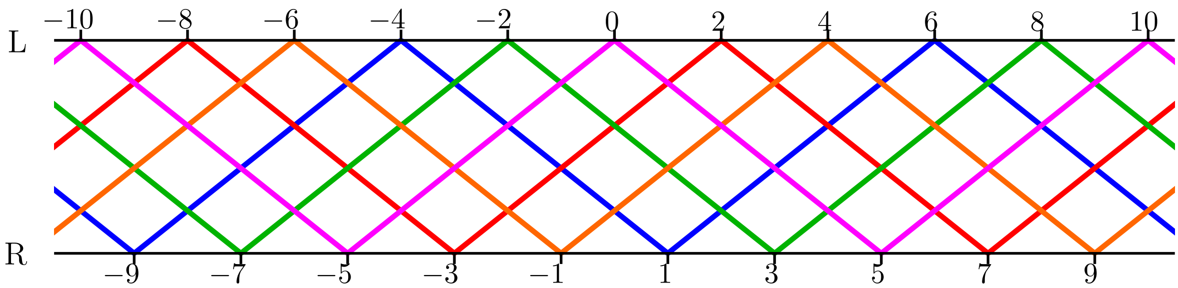


More constant patterns:

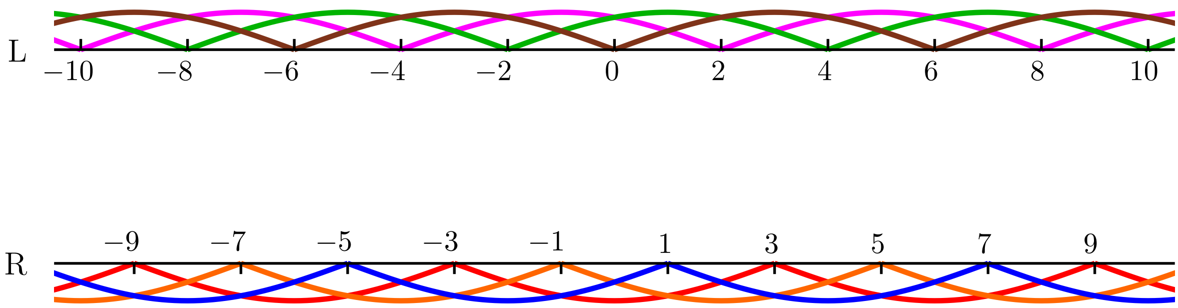
$c = 4$



$c = 5$

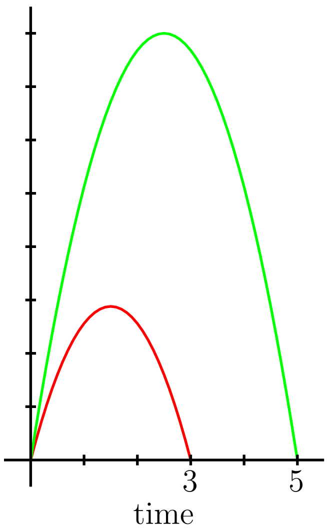


$c = 6$



Practical matters:

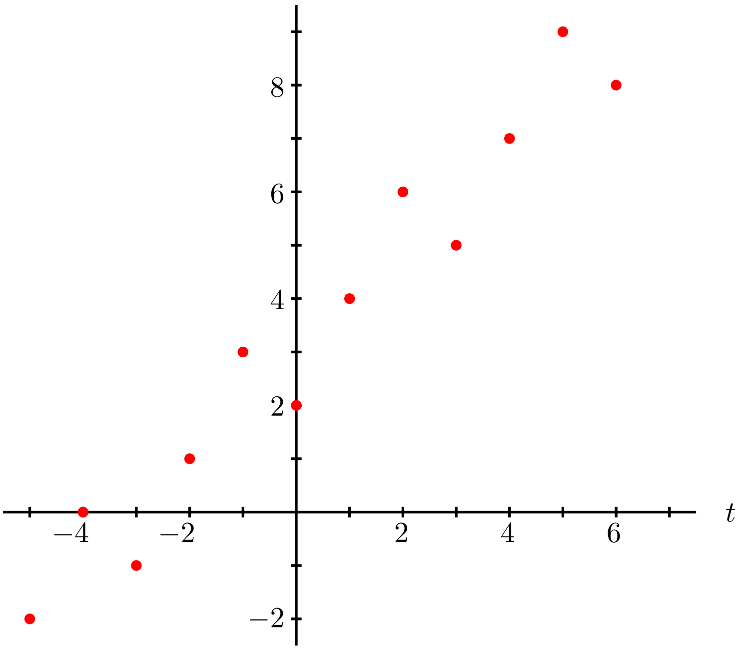
- Since the hands alternate, a throw goes to the opposite hand if h_t is odd, and to the same hand if h_t is even.
- The number h_t measures the ball's time in the air. Height is proportional to the square of flight time, so a $h_t = 5$ throw is about $(5/3)^2$ times as high as a $h_t = 3$ throw.
- In practice, a $h_t = 2$ throw is a held ball.



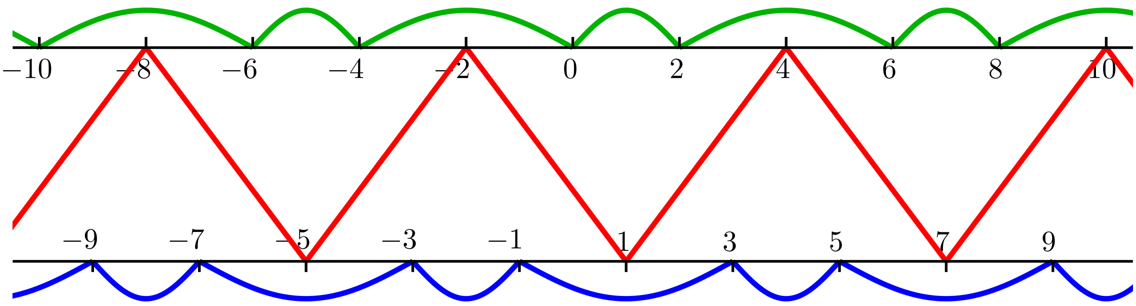
Example:

$$h_t = \ldots 2, 3, 4, 2, 3, 4, 2, 3, 4, \ldots$$

t	h_t	$t + h_t$
$\vdots$	$\vdots$	$\vdots$
-3	2	-1
-2	3	1
-1	4	3
0	2	2
1	3	4
2	4	6
3	2	5
4	3	7
5	4	9
$\vdots$	$\vdots$	$\vdots$



The map $t \mapsto t + h_t$ appears to be bijective.



Definition: A sequence

$$\ldots, h_{-2}, h_{-1}, h_0, h_1, h_2, \ldots$$

is called *n-periodic* if $h_{t+n} = h_t$ for all t.

Example: The sequence

$$\ldots, 2, 3, 4, 2, 3, 4, 2, 3, 4, \ldots$$

is 3-periodic.

It is also 6-periodic, 9-periodic, 12-periodic, and so on.

Definition: A sequence

$$\ldots, h_{-2}, h_{-1}, h_0, h_1, h_2, \ldots$$

is *exactly n-periodic* if it is n-periodic and is not m-periodic for any positive integer $m < n$.

Proposition: Let $\{h_t\}_{t\in\mathbb{Z}}$ be an n-periodic sequence. Then the map

$$t \ \mapsto \ t + h_t$$

is a bijection from $\mathbb{Z}$ to $\mathbb{Z}$ if and only if the map

$$t \ \mapsto \ (t + h_t) \bmod n$$

is a bijection from $\{0, 1, \ldots, n-1\}$ to $\{0, 1, \ldots, n-1\}$.

Proof: Exercise.

Corollary: A sequence $h_0, h_1, h_2, \ldots, h_{n-1}$ defines an n-periodic juggling pattern if and only if the set

$$\{0 + h_0, 1 + h_1, 2 + h_2, \ldots, (n-1) + h_{n-1}\},$$

reduced modulo n, gives the set

$$\{0, 1, 2, \ldots, n-1\}$$

Remark: We now have an easy way to check whether a periodic sequence is a juggling pattern.

Examples:

Period 3

$$\begin{array}{rccc} h_t: & 2 & 4 & 5 \\ t: & 0 & 1 & 2 \\ \hline & 2 & 2 & 1 \end{array} \text{ –No}$$

$$\begin{array}{rccc} h_t: & 1 & 5 & 3 \\ t: & 0 & 1 & 2 \\ \hline & 1 & 0 & 2 \end{array} \text{ –Yes}$$

Period 4

$$\begin{array}{rcccc} h_t: & 1 & 4 & 1 & 6 \\ t: & 0 & 1 & 2 & 3 \\ \hline & 1 & 1 & 3 & 1 \end{array} \text{ –No}$$

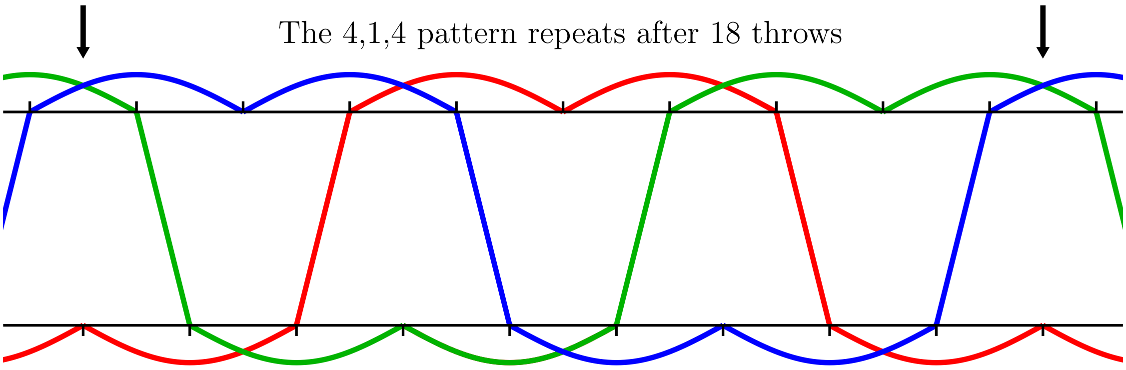
$$\begin{array}{rcccc} h_t: & 3 & 4 & 2 & 3 \\ t: & 0 & 1 & 2 & 3 \\ \hline & 3 & 1 & 0 & 2 \end{array} \text{ –Yes}$$

Theorem: The number of balls used in a periodic juggling pattern $h_0, h_1, \ldots, h_{n-1}$ is

$$\frac{1}{n}\sum_{k=0}^{n-1} h_k$$

Proof:

Choose p large enough so that at the p^{th} repetition of the pattern, all the balls are in their starting places.



Let $\mathcal{B}$ denote the set of balls. Let $M = \sum_{b\in\mathcal{B}} \begin{pmatrix} \text{amount of time ball} \\ b \text{ is in the air} \\ \text{through } p \text{ periods} \end{pmatrix}$.

In theory, every ball is in the air through every tick of the clock, and there are pn ticks of the clock in p periods, so

$$M = (\text{number of balls}) \times (pn) \qquad (1)$$

$$M = (\text{number of balls}) \times (pn) \tag{1}$$

On the other hand, we can calculate M by adding up the heights of all the throws through p periods, so

$$M = p\sum_{k=0}^{n-1} h_k \tag{2}$$

Combining (1) and (2), we have

$$pn \times (\text{number of balls}) = p\sum_{k=0}^{n-1} h_k$$

from which the result follows.

Example:

The $3, 4, 2, 3$ pattern uses $\frac{3+4+2+3}{4} = 3$ balls

The $3, 4, 5$ pattern uses $\frac{3+4+5}{3} = 4$ balls

Question: How many 2-periodic three-ball patterns are there?

Answer: The 2-periodic three-ball patterns are

1 5   2 4   3 3

(Only 1 5 and 2 4 are exactly 2-periodic.)

Question: How many 2-periodic four-ball patterns are there?

Answer: The 2-periodic four-ball patterns are

1 7   2 6   3 5   4 4

Question: In general, how many n-periodic b-ball patterns are there?

Theorem: (Buhler et. al. 1994)
The number of n-periodic b-ball juggling patterns is $(b+1)^n - b^n$.

Remarks: In this theorem, cyclic shifts are counted as distinct patterns, and it counts n-period patterns, rather than just the exact n-periodic ones.

Question: How many 3-periodic three-ball juggling patterns are there?

Answer: The theorem says that there are

$$4^3 - 3^3 = 64 - 27 = 37$$

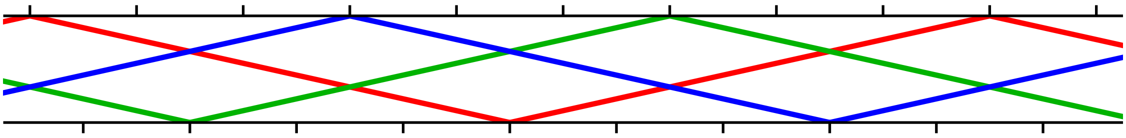
three-ball 3-periodic juggling patterns. One of them is $\ldots, 3, 3, 3, \ldots$, which isn't exactly 3-periodic.

Of the remaining 36, each one gets counted three times, because it has three cyclic permutations (for example, $(4,1,4)$, $(1,4,4)$, and $(4,4,1)$ are really all the same pattern). So we have 12 distinct three-ball patterns that are exactly 3-periodic.

The twelve exactly 3-periodic three-ball juggling patterns

0 0 9	0 1 8	0 3 6
0 4 5	0 6 3	0 7 2
1 1 7	1 2 6	1 4 4
1 5 3	2 2 5	2 3 4

009:



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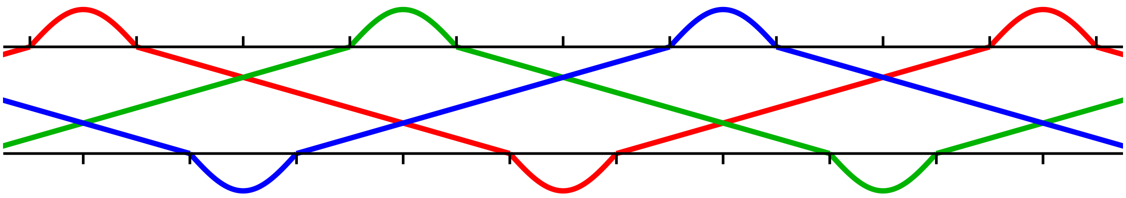

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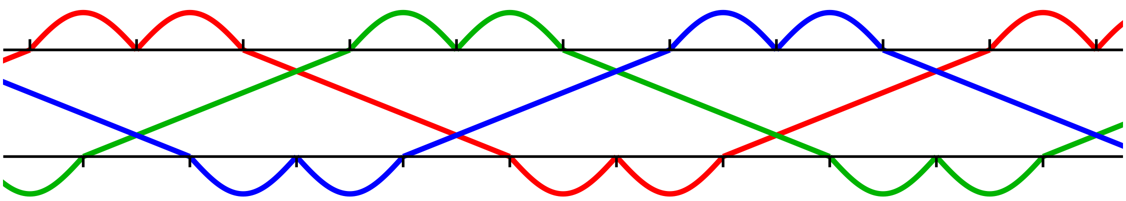

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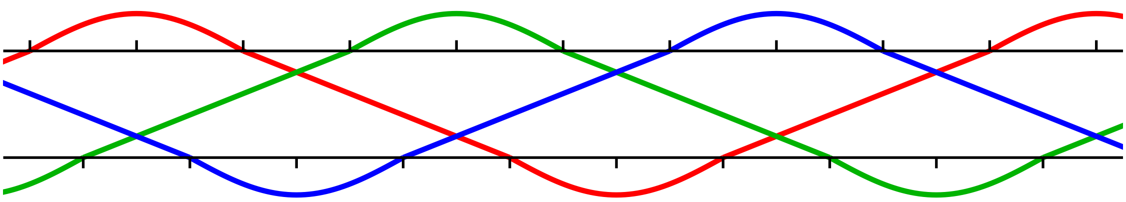

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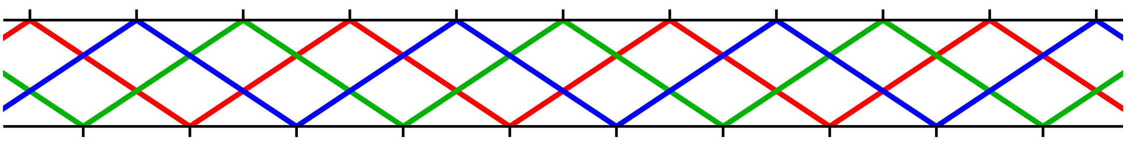

Closing questions:

- How can we classify and recognize patterns that are just time-dilations of other patterns?
- How many b-ball n-periodic patterns are there that are truly distinct?
- Some patterns look like small embellishments of other patterns. Is there a sensible way to "factor" juggling patterns? What are the primes?
- What does all this have to do with braid theory?