Sharing a Secret
in Plain Sight

Gregory Quenell

The Setting: Alice and Bob want to have a private conversation using email or texting.

Alice

Bob

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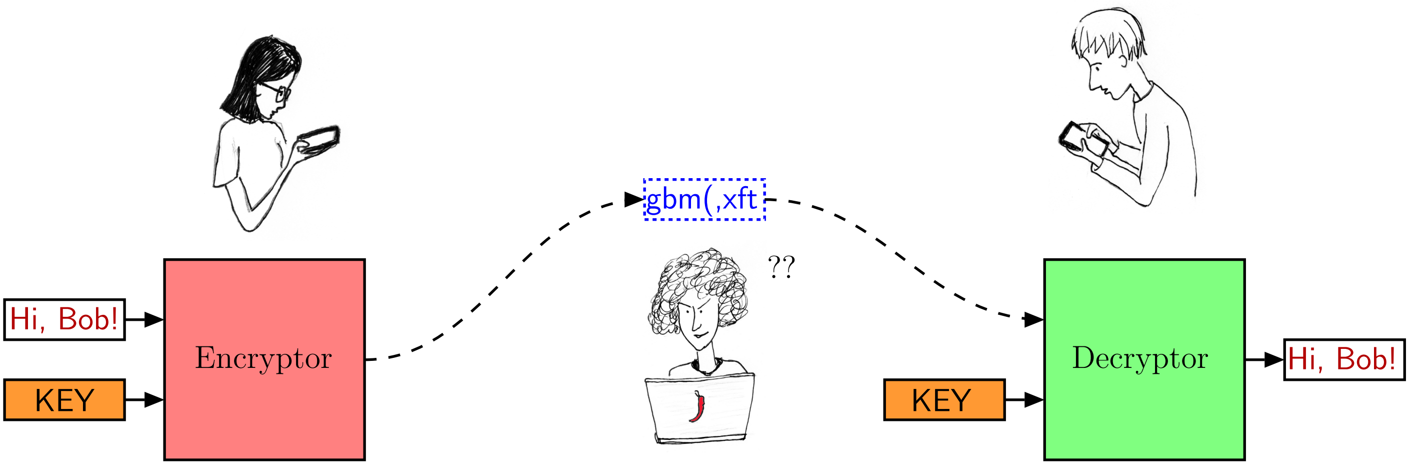
The Problem: These media are insecure. Anyone can listen in by intercepting wifi packets.

The Solution: Encryption. In a *symmetric-key* or *shared-key* encryption scheme, Bob and Alice share a secret key (KEY) that Eve doesn't know.

Alice feeds KEY and her message into an Encryptor, which "locks" the message so that Eve can't read it.

Bob uses his copy of KEY in the Decryptor to "unlock" and read the message.



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More realistically: Can Alice and Bob use the insecure channel, where Eve can intercept everything, to

- come up with a key that they both know, and
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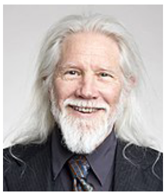
Surprisingly, the answer is yes.

The Diffie-Hellman Key Exchange Protocol

. . . allows two people who have no prior knowledge of one another to establish a

shared secret key

while communicating over an insecure channel.



The Diffie-Hellman protocol was first described in a 1976 paper by Whitfield Diffie and Martin Hellman.

Similar systems had previously been described by American cryptographer Ralph Merkle, and in classified research at the Government Communications Headquarters (GCHQ) in England.

Exponentiation in the ring $\mathbb{Z}_p$

For $g \in \mathbb{Z}_p$ and a positive integer k, we can compute $g^k \in \mathbb{Z}_p$ by repeatedly multiplying and reducing modulo p.

Example: 3^k in $\mathbb{Z}_7$.

$$
\begin{array}{rcccccccl}
 & & 3^1 & = & 3 & & & \equiv & 3 \pmod 7 \\
 & & 3^2 & = & 3 \times 3 & = & 9 & \equiv & 2 \pmod 7 \\
3^3 & = & 3 \times 3^2 & = & 3 \times 2 & = & 6 & \equiv & 6 \pmod 7 \\
3^4 & = & 3 \times 3^3 & = & 3 \times 6 & = & 18 & \equiv & 4 \pmod 7 \\
3^5 & = & 3 \times 3^4 & = & 3 \times 4 & = & 12 & \equiv & 5 \pmod 7 \\
3^6 & = & 3 \times 3^5 & = & 3 \times 5 & = & 15 & \equiv & 1 \pmod 7 \\
3^7 & = & 3 \times 3^6 & = & 3 \times 1 & = & 3 & \equiv & 3 \pmod 7 \\
\vdots & & & & \vdots & & & & \vdots
\end{array}
$$

The powers of 3 in the system $\mathbb{Z}_7$ look like this:

n	0	1	2	3	4	5	6	7	8	$\cdots$
3^n	1	3	2	6	4	5	1	3	2	$\cdots$

Exponentiation in the ring $\mathbb{Z}_p$

There's a much faster method, called the square-and-multiply method.

Example: 2^{68} in $\mathbb{Z}_{101}$.

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\begin{array}{lllllllll}
2^2 & & & & & = & 4 & \equiv & 4 \pmod{101} \\
2^4 & = & (2^2)^2 & = & 4^2 & = & 16 & \equiv & 16 \pmod{101} \\
2^8 & = & (2^4)^2 & = & 16^2 & = & 256 & \equiv & 54 \pmod{101} \\
2^{16} & = & (2^8)^2 & = & 54^2 & = & 2916 & \equiv & 88 \pmod{101} \\
2^{32} & = & (2^{16})^2 & = & 88^2 & = & 7744 & \equiv & 68 \pmod{101} \\
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Now $2^{68} = 2^{64+4} = 2^{64} \times 2^4$, so we get

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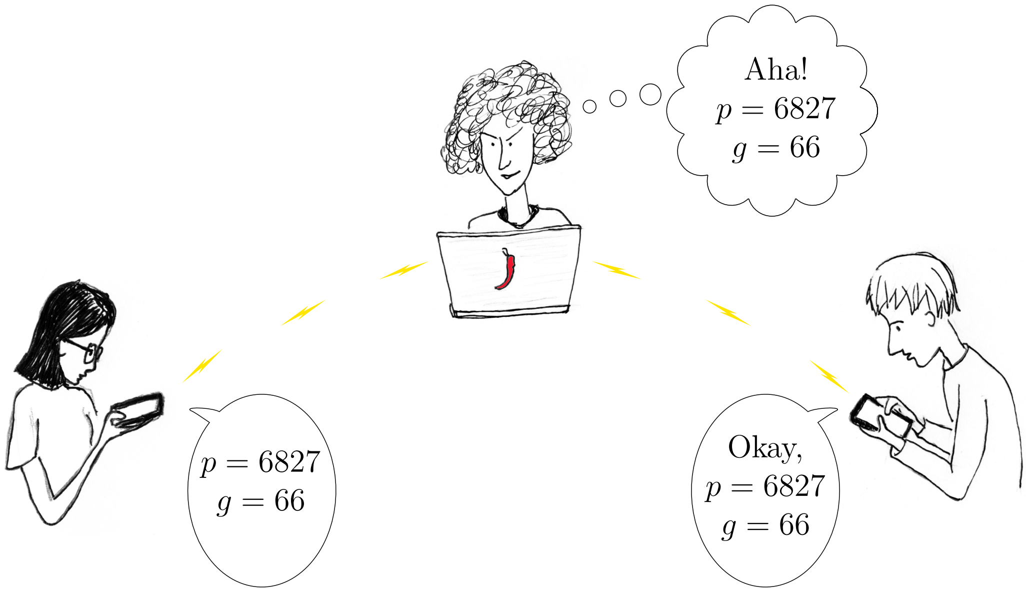
Cost: seven multiplications (with reductions modulo 101).

Alice and Bob

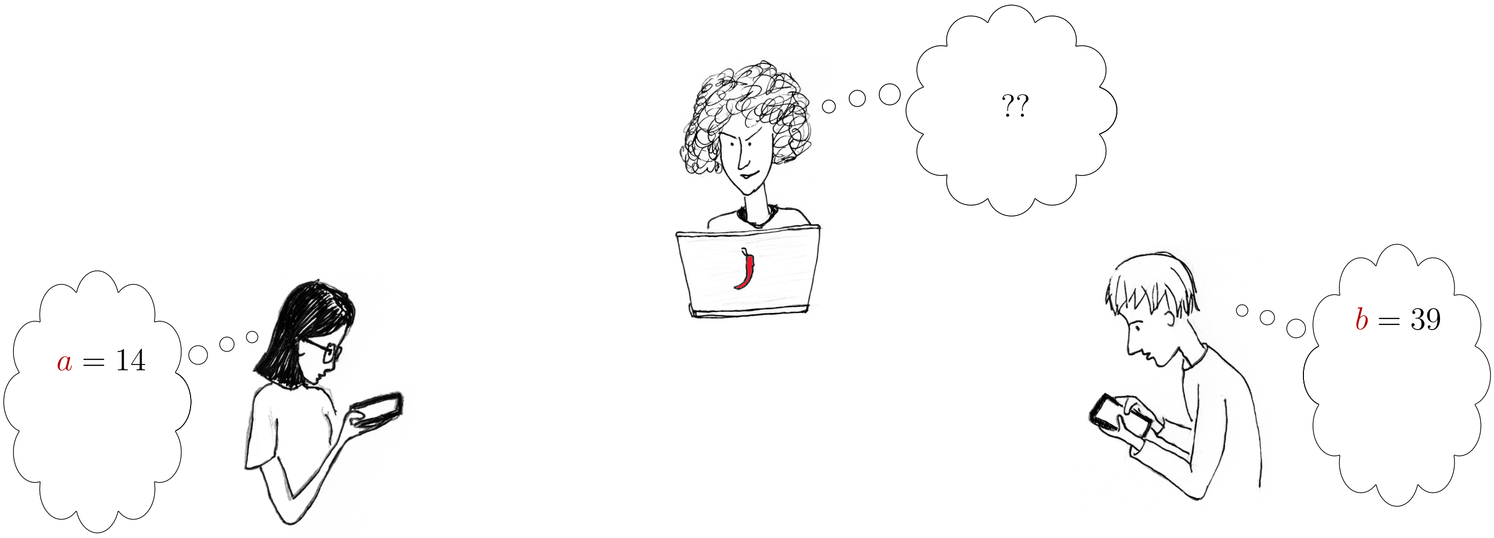
- Alice selects a prime p and a base g, and sends these to Bob.
- Eve listens in; she now knows the values of p and g.



Alice and Bob

$(p = 6827; g = 66)$

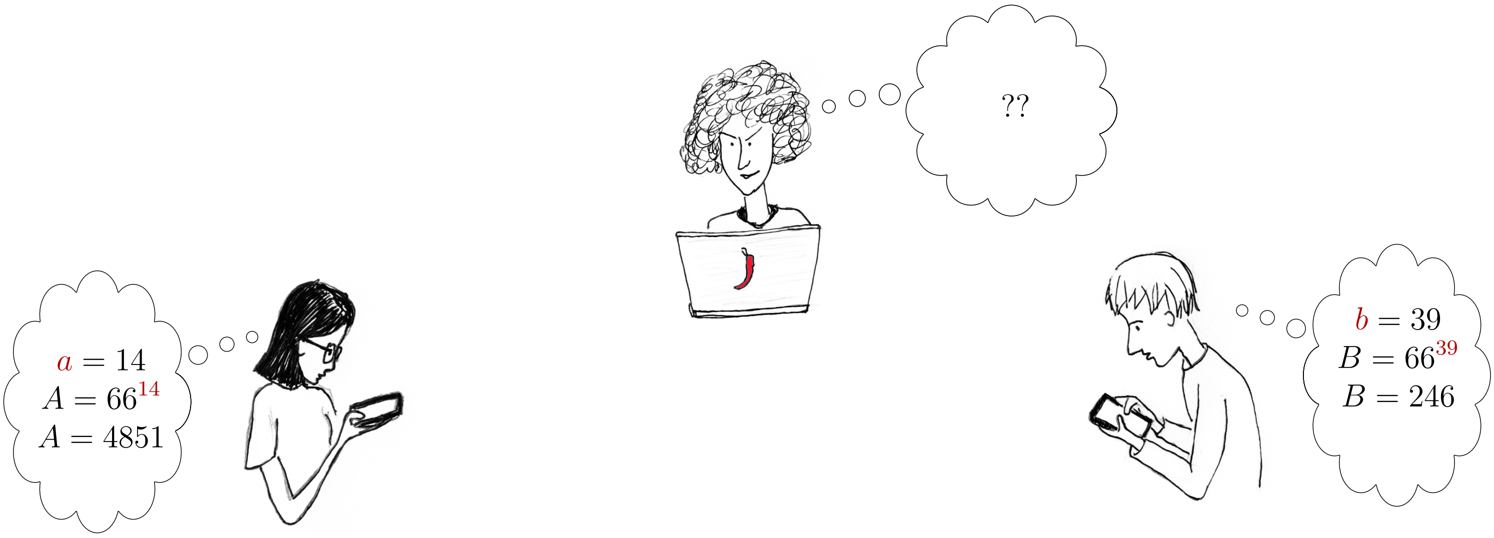
- Alice chooses a secret exponent a, and doesn't tell anyone.
- Bob chooses a secret exponent b, and doesn't tell anyone.



Alice and Bob

$(p = 6827;\ g = 66)$

- Alice uses square-and-multiply to compute $A = g^{a}$.
- Bob uses square-and-multiply to compute $B = g^{b}$.



Alice and Bob

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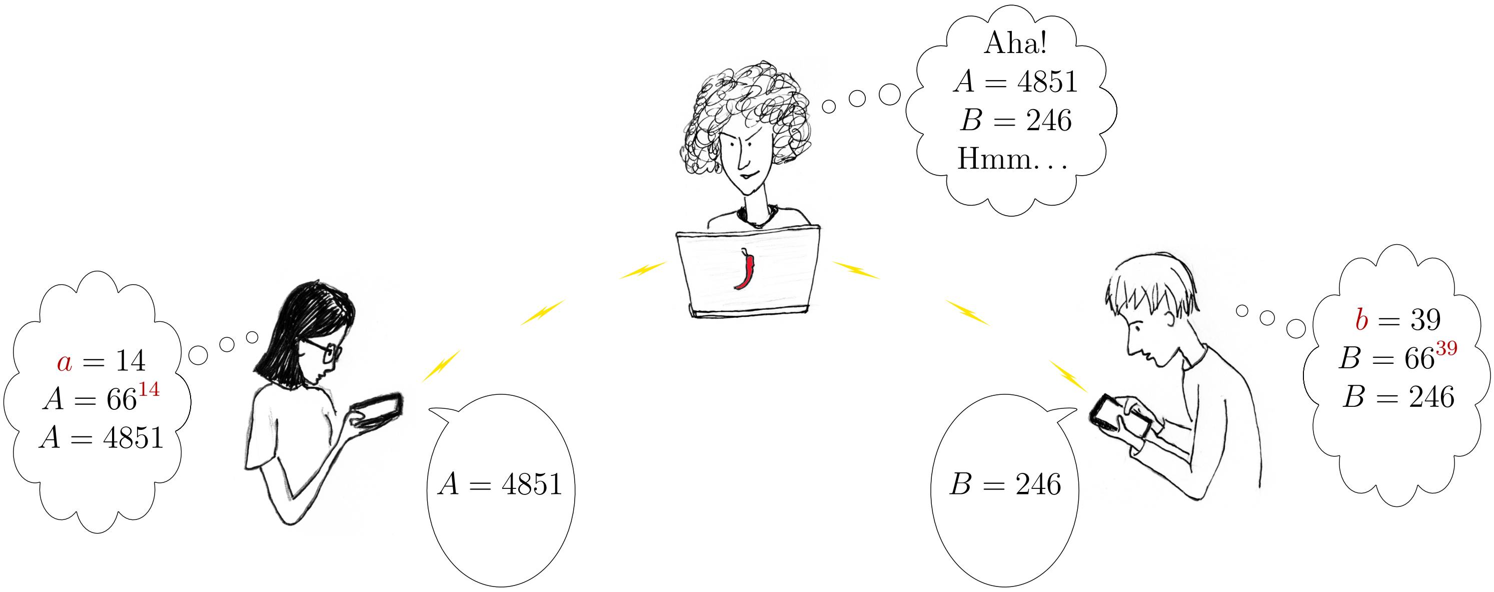
- Alice sends A to Bob, and Bob sends B to Alice.
- Eve now knows the values of g, $A = g^a$, and $B = g^b$.



Alice and Bob

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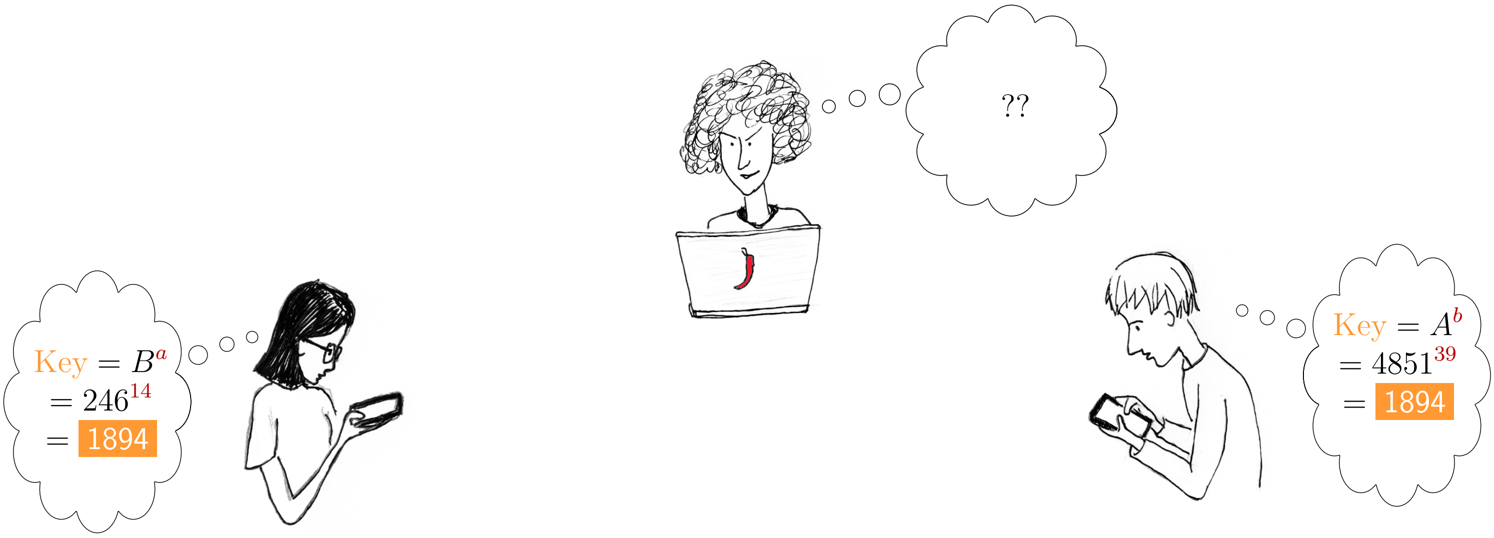
- Bob now computes A^b, which is equal to $(g^a)^b$.
- Alice now computes B^a, which is equal to $(g^b)^a$.

By the laws of exponents, $(g^a)^b = g^{ab} = (g^b)^a$, so Bob and Alice have the *same number*.



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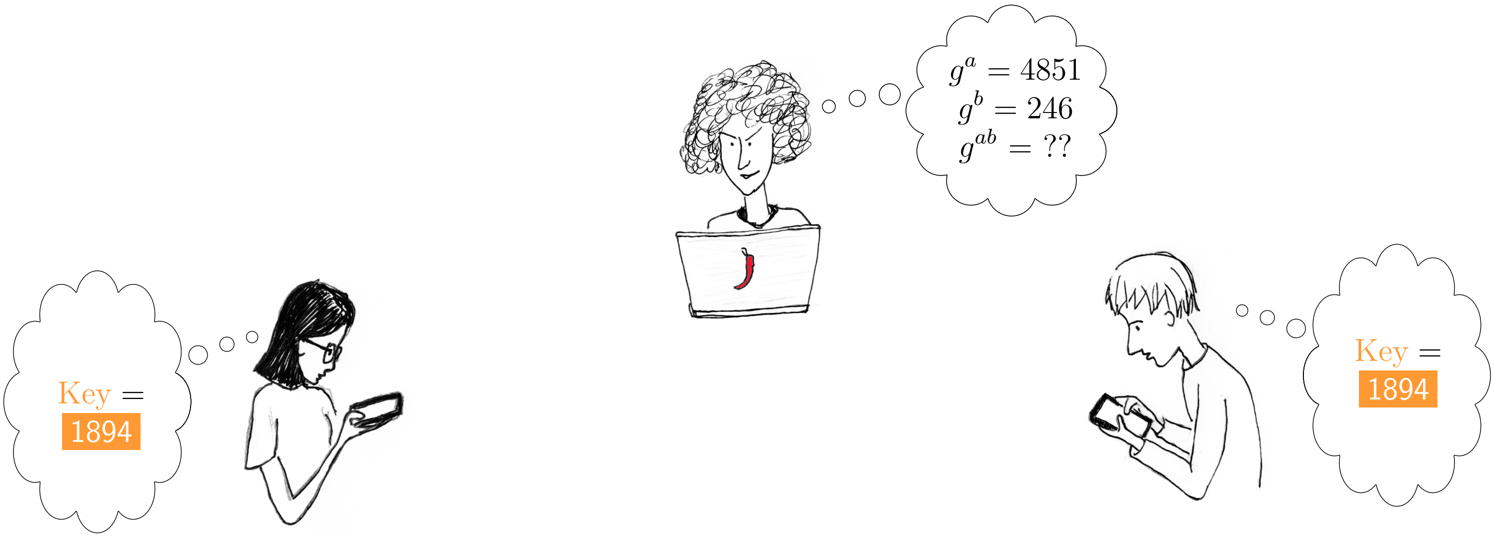
- Alice and Bob, each using square-and-multiply twice, have independently calculated g^{ab}.
- Eve, by listening in, has learned g, g^a and g^b, but has no good way to calculate g^{ab} from this information.



Security

The secrecy of the shared key g^{ab} relies on the

Computational Diffie-Hellman Assumption

CDH: Let g be a generator of a cyclic group G. Given generic elements g^a and g^b, Eve has no efficient algorithm for finding g^{ab}:

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Belief: For certain cyclic groups, Eve's best approach is to find either a or b, the *discrete logarithms* of g^a and g^b.

> DLP (the Discrete Logarithm Problem): Let g be a generator of a cyclic group G. Given a generic element g^a, find a.

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Result: In our toy example, $|G| = 6826$, so Bob and Alice can find their shared key g^{ab} using no more than

$$2\log_2 6826 \approx 26 \text{ multiplications.}$$

In order to find a and "break in", Eve's best algorithm would require at least

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In the toy example, Eve's task is trivial. There is no secrecy here. However . . .

Security

$\|G\|$	$\sqrt{\|G\|}$	$\log_2 \|G\|$	Time at 10^9/second Alice/Bob	Eve
10^5	300	17	4.6 μs	87 μs
10^{15}	3×10^7	33	120 μs	78 sec
10^{25}	3×10^{12}	83	570 μs	250 days
10^{35}	3×10^{17}	116	1.6 ms	1.4×10^5 yr
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(NIST recommends that $\lvert G\rvert$ should be at least 10^{68}, *and* that G should be hidden inside a bigger group with order at least 10^{616}.)

References

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