String Art and Calculus

(and Games with Envelopes)

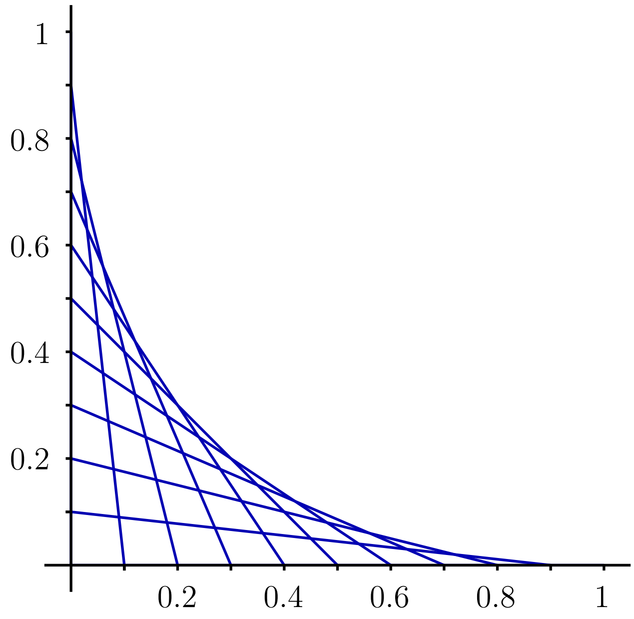
Gregory Quenell

First example

Draw line segments connecting

$$(0, x) \quad \text{with} \quad (1 - x, 0)$$

for $x = 0.1,\ 0.2,\ \ldots,\ 0.9.$



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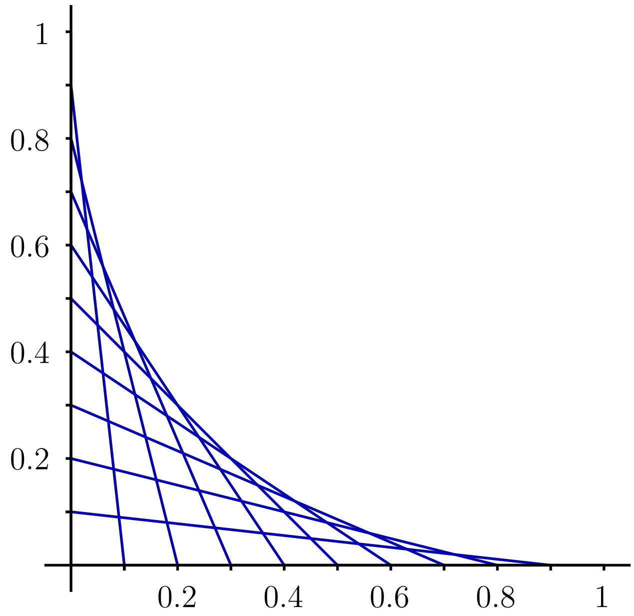

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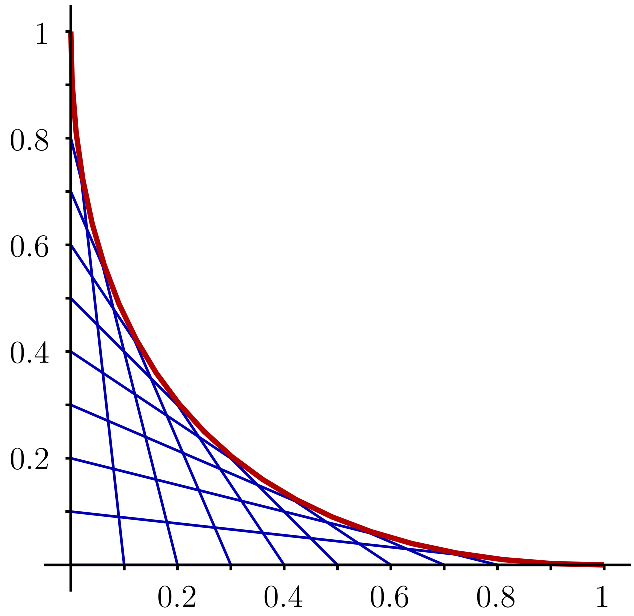

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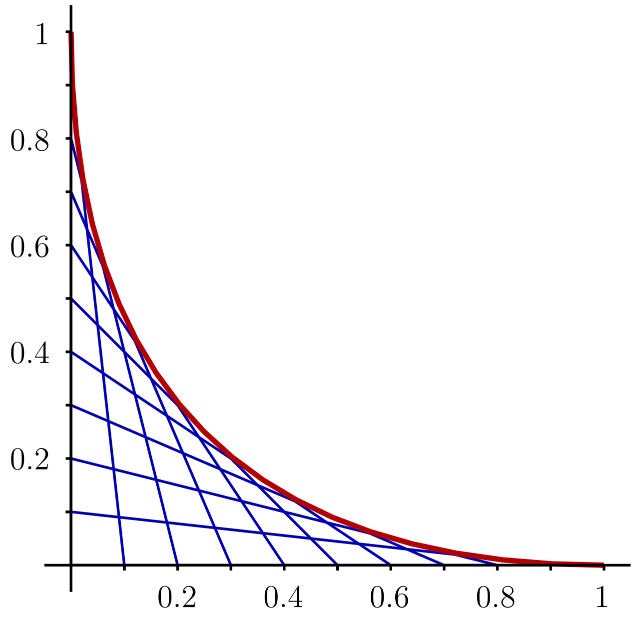
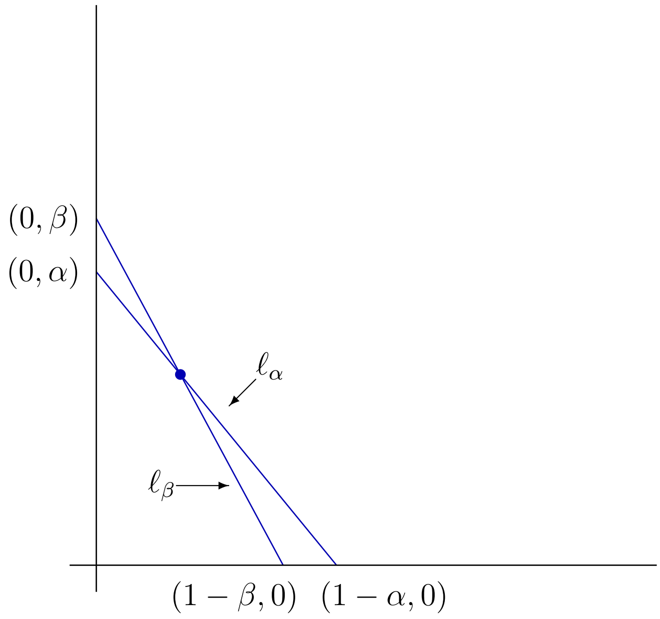


What curve is it?

Finding the envelope

For each $\alpha \in [0,1]$, let ℓ_α be the line segment connecting

$$(0,\alpha) \ \text{ with } \ (1-\alpha,0).$$



Finding the envelope

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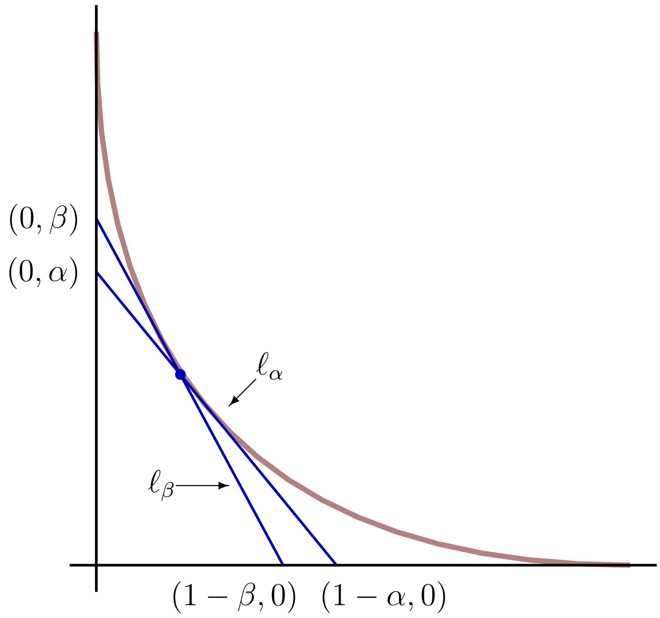
If α and β are close together, then the intersection point of ℓ_α and ℓ_β is close to a point on the curve.



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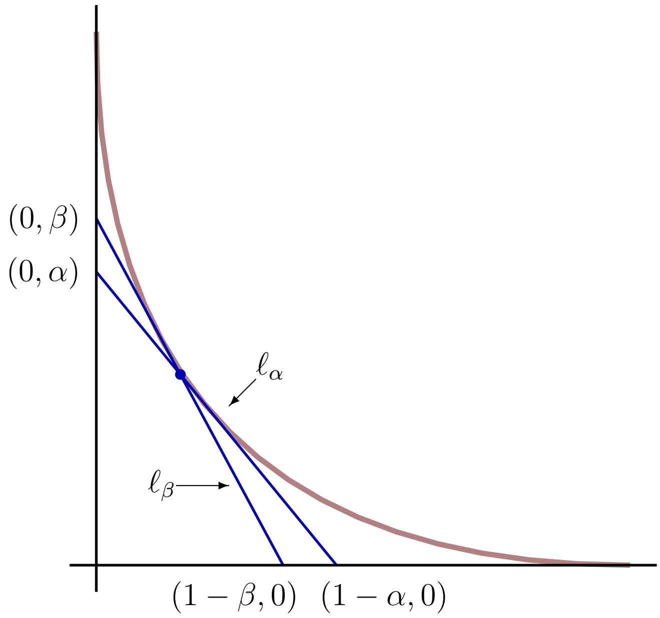


Exercise: For $\alpha \neq \beta$, the segments ℓ_α and ℓ_β intersect at the point

$$(\alpha\beta, (1 - \alpha)(1 - \beta)).$$

Finding the envelope

As $\beta \to \alpha$, the point

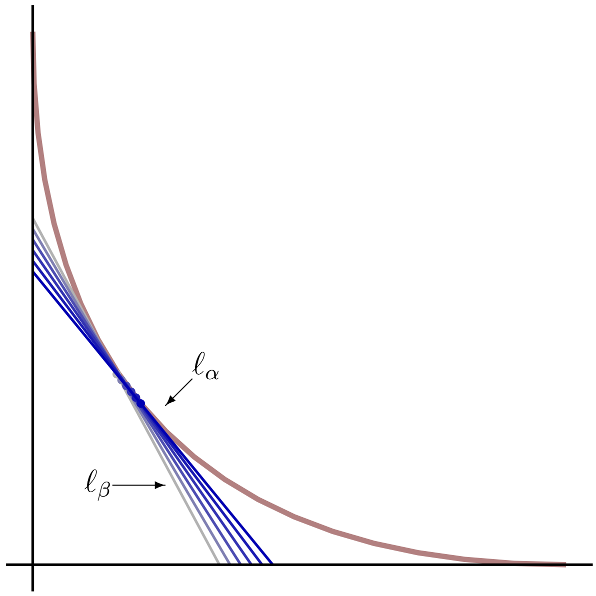
$$(\alpha\beta, (1-\alpha)(1-\beta))$$

approaches a point on the curve.

Thus, each point on the curve has the form

$$\lim_{\beta\to\alpha} (\alpha\beta, (1-\alpha)(1-\beta))$$

for some α.



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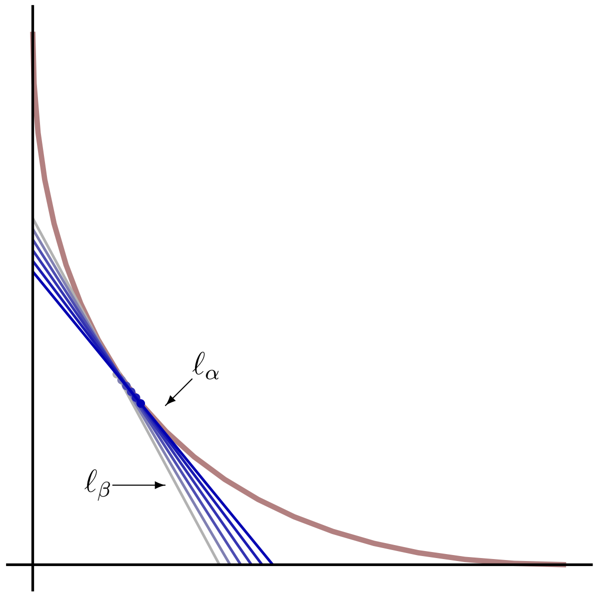


Thus, each point on the curve has the form

$$\lim_{\beta\to\alpha}(\alpha\beta, (1-\alpha)(1-\beta))$$

for some α.

This is an easy limit, and we get the parametrization

$$(\alpha^2, (1-\alpha)^2), \quad 0 \le \alpha \le 1$$

for our envelope curve.

Finding the envelope

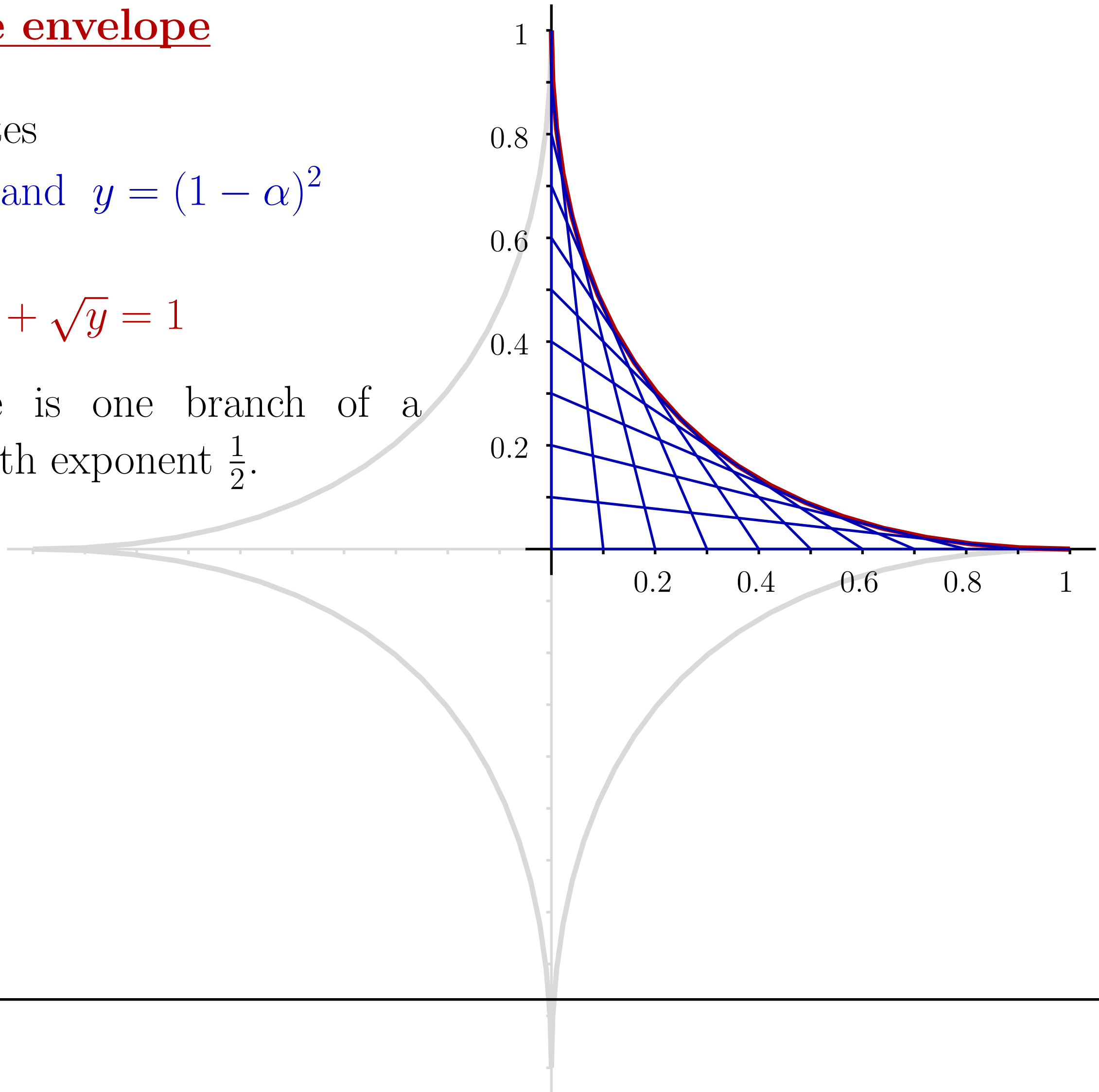
The coordinates

$$x = \alpha^2 \text{ and } y = (1-\alpha)^2$$

satisfy

$$\sqrt{x} + \sqrt{y} = 1$$

so our curve is one branch of a *hypocircle* with exponent $\frac{1}{2}$.



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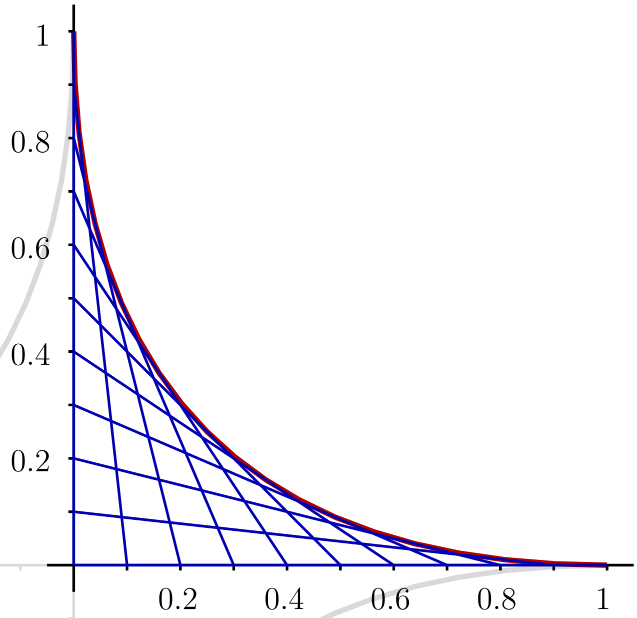


Stewart 4/e, p. 191, problem 38 says

> *"Show that the sum of the x- and y-intercepts of any tangent line to the curve $\sqrt{x} + \sqrt{y} = \sqrt{c}$ is equal to c."*

Parabolas

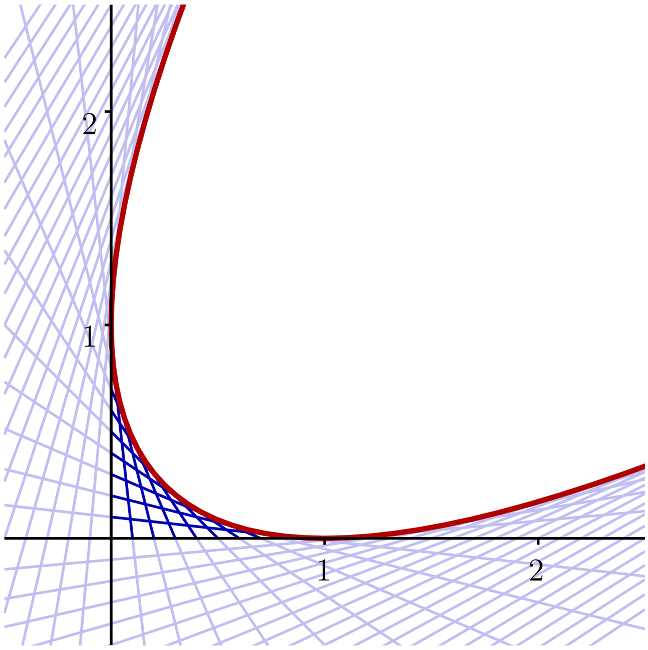
The coordinates

$$x = \alpha^2 \text{ and } y = (1-\alpha)^2$$

also satisfy

$$2(x+y) = (x-y)^2 + 1$$

Our envelope curve is part of a parabola, tangent to the coordinate axes at $(1, 0)$ and $(0, 1)$.



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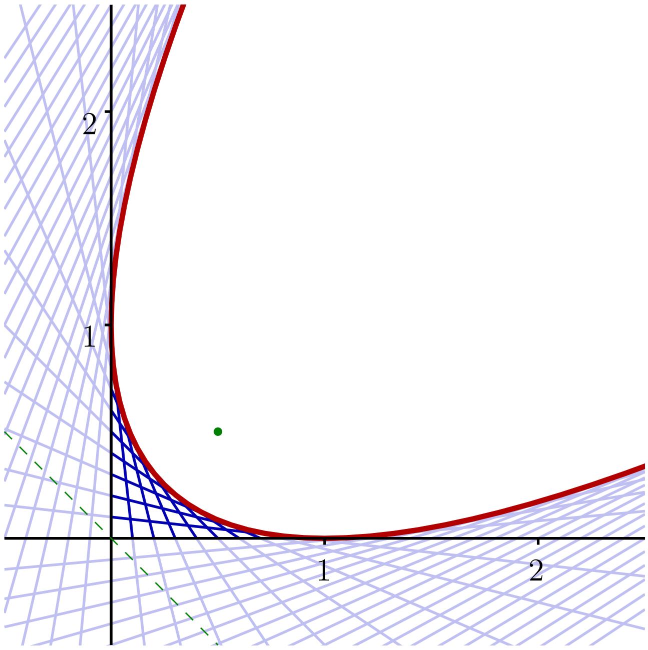


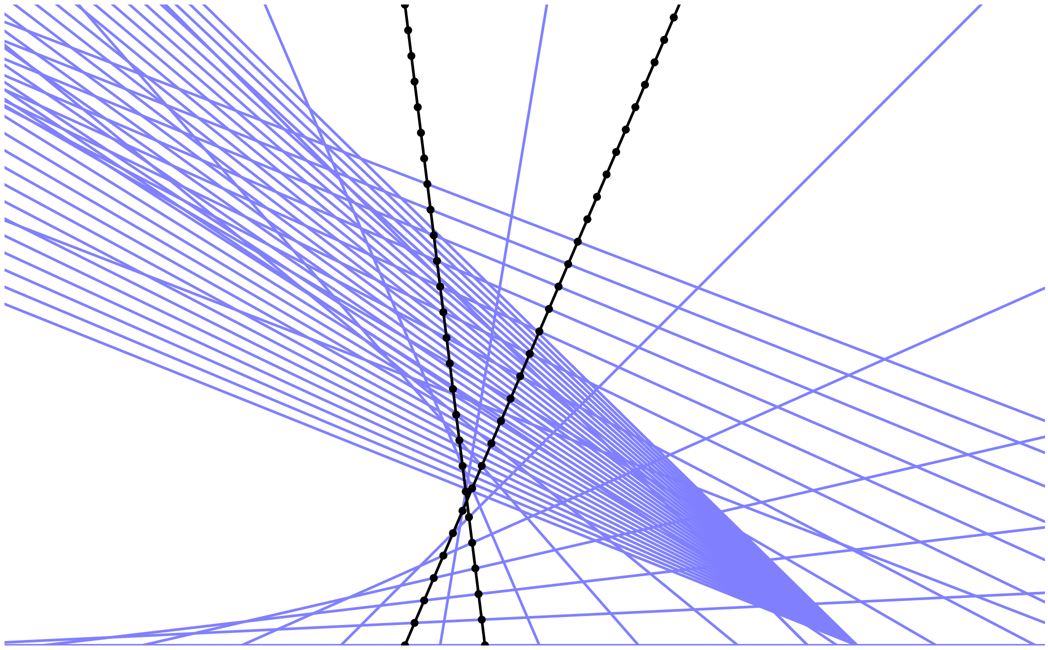
Our envelope curve is part of a parabola, tangent to the coordinate axes at $(1, 0)$ and $(0, 1)$.

In the classical theory of conic sections, our envelope has

$$\text{focus} \quad \left(\tfrac{1}{2}, \tfrac{1}{2}\right) \quad \text{and directrix} \quad x + y = 0.$$

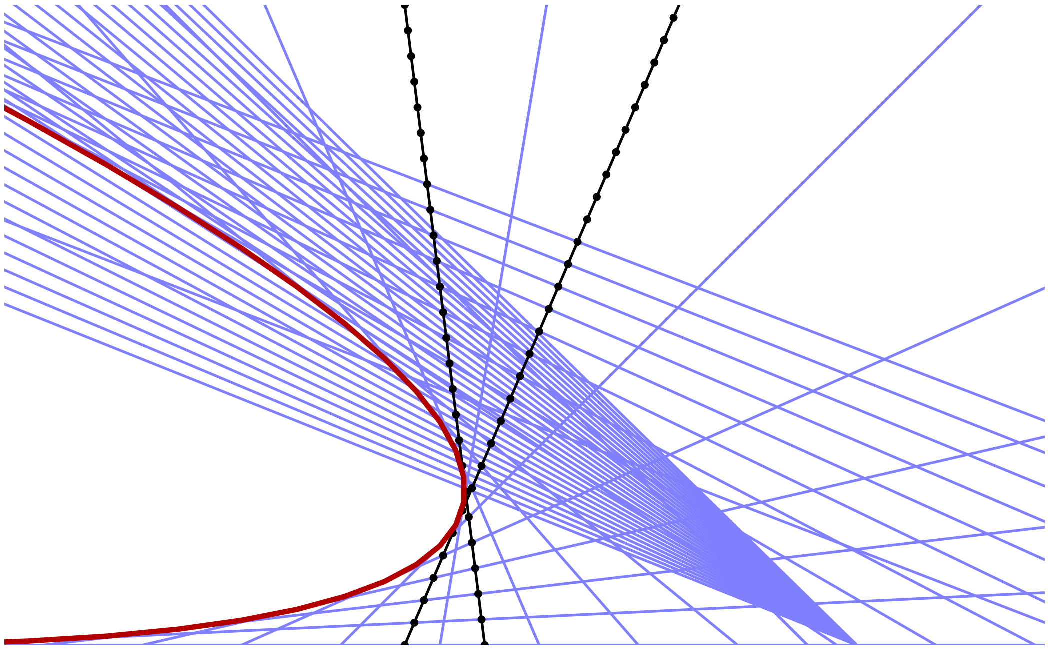
An easy generalization

Pick equally-spaced points along (almost) any two lines, and do the same thing. You get an image of our parabola under a linear transformation.



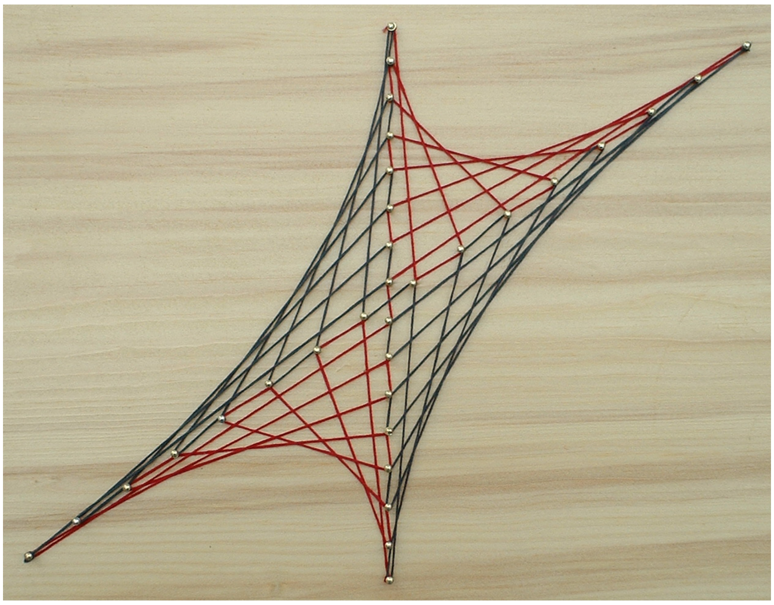
An easy generalization

Pick equally-spaced points along (almost) any two lines, and do the same thing. You get an image of our parabola under a linear transformation. It's another parabola.



Application: String art

Drive nails at equal intervals along two lines, and connect the nails with decorative string.



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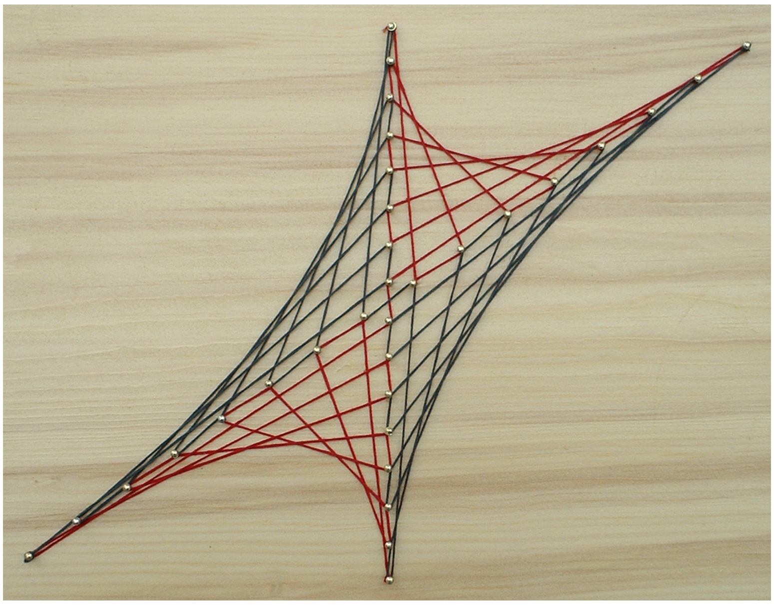

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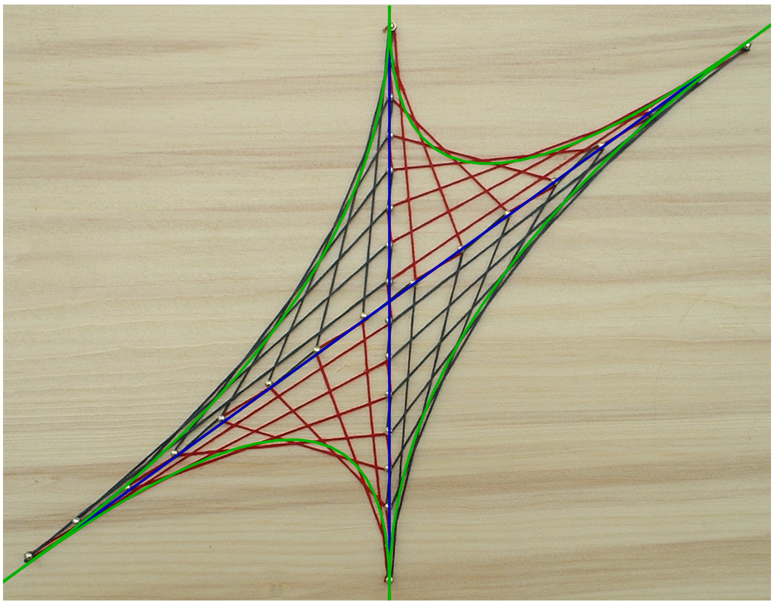


… envelope curves that lie on parabolas tangent to the nailing lines.

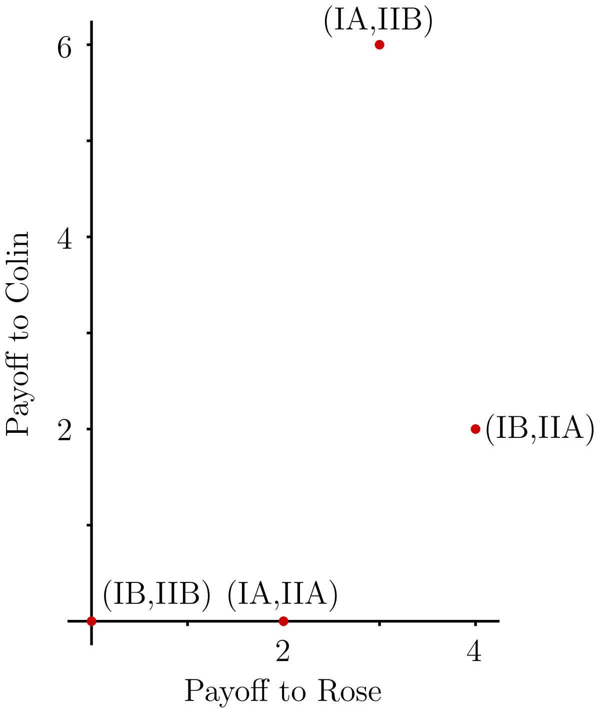
Application: game theory

Consider a two-person, non-zero-sum game in which each player has two strategies.

		Colin	
		IIA	IIB
Rose	IA	$(2,0)$	$(3,6)$
	IB	$(4,2)$	$(0,0)$

Such a game has four possible payoffs. We list them in a *payoff matrix*.

We can show the payoffs to Rose and Colin as points in the *payoff plane*.



Game theory assumptions

We assume each player adopts a randomized *mixed strategy*:

- Rose plays IA with probability p and IB with probability $1-p$.
- Colin plays IIA with probability q and IIB with probability $1-q$

		Colin	
		IIA	IIB
Rose	IA	$(2,0)$	$(3,6)$
	IB	$(4,2)$	$(0,0)$

The *expected payoff* is then

$$pq(2,0)+p(1-q)(3,6)+(1-p)q(4,2)+(1-p)(1-q)(0,0)$$

or

$$p\left[q(2,0)+(1-q)(3,6)\right]+(1-p)\left[q(4,2)+(1-q)(0,0)\right]$$

or

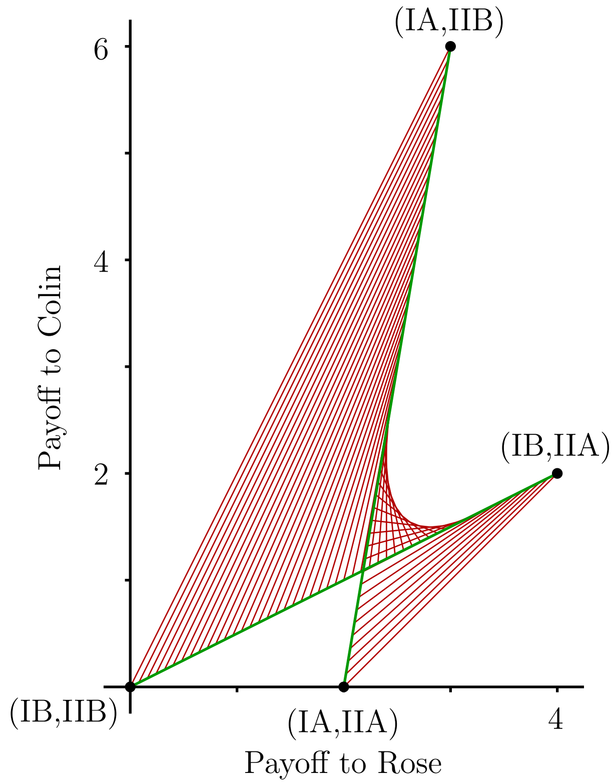
$$q\left[p(2,0)+(1-p)(4,2)\right]+(1-q)\left[p(3,6)+(1-p)(0,0)\right]$$

Possible expected payoffs

Each value of q determines one point on the line from $(2, 0)$ to $(3, 6)$ and one point on the line from $(4, 2)$ to $(0, 0)$.

Then p is the parameter for a line segment between these points.

$$p\,[q(2,0) + (1-q)(3,6)] \\ +(1-p)\,[q(4,2) + (1-q)(0,0)]$$

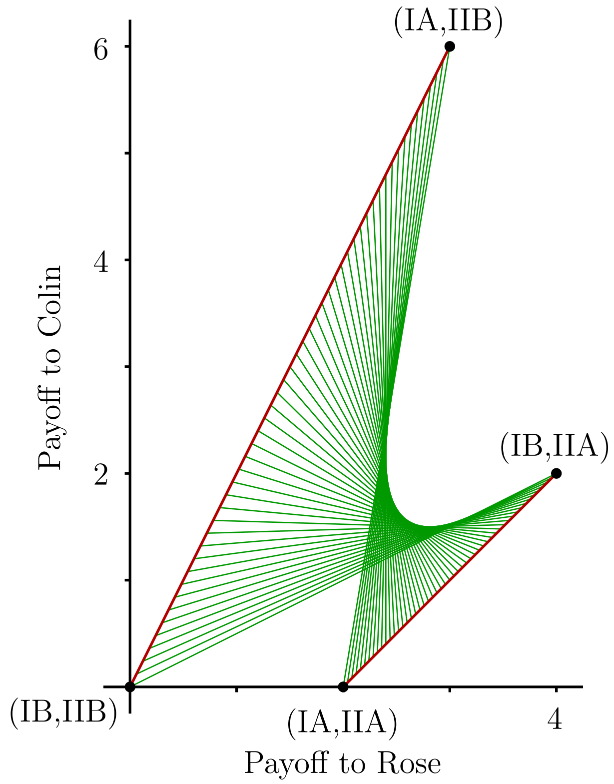


Possible expected payoffs

Alternatively, each value of p determines one point on the line from $(2, 0)$ to $(4, 2)$ and one point on the line from $(3, 6)$ to $(0, 0)$.

Then q is the parameter for a line segment between these points.

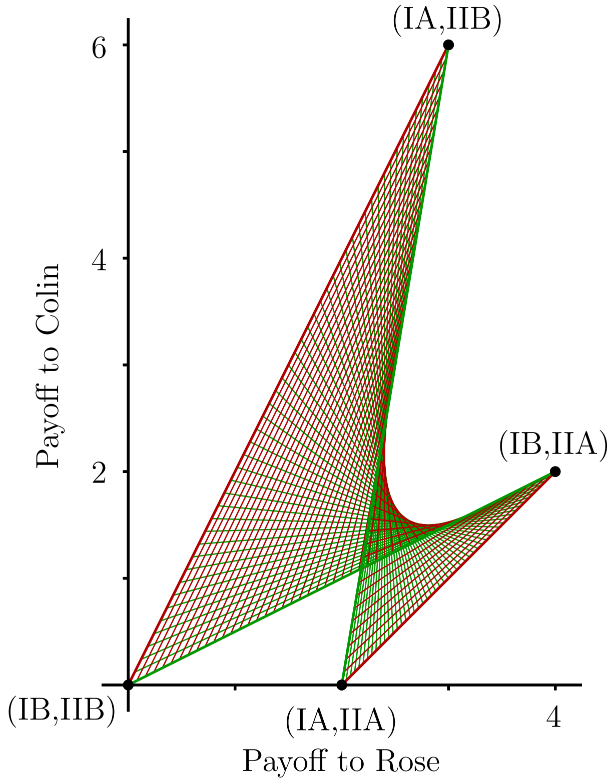
$$q\,[p(2,0) + (1-p)(4,2)] \\ +(1-q)\,[p(3,6) + (1-p)(0,0)]$$



Possible expected payoffs

Either way, the expected payoff is contained in a region bounded by four lines and a parabolic envelope curve.

If the game is played a large number of times and the average payoff converges to a point outside this region, then the players' randomizing devices are not independent.



This could be due to collusion, espionage, or maybe just poor random-number generators.

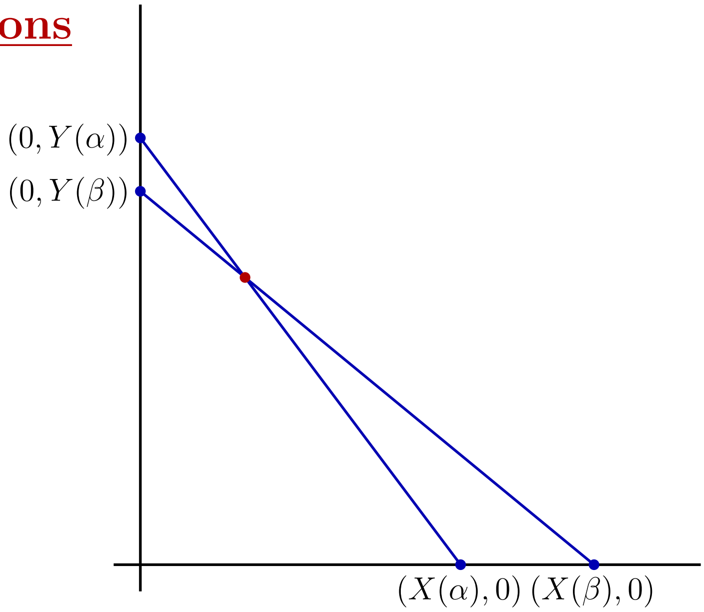
Generalization: spacing functions

Draw line segments ℓ_α connecting

$$(X(\alpha), 0) \ \text{ with } \ (0, Y(\alpha))$$

for arbitrary differentiable functions X and Y.

These are "spacing functions".



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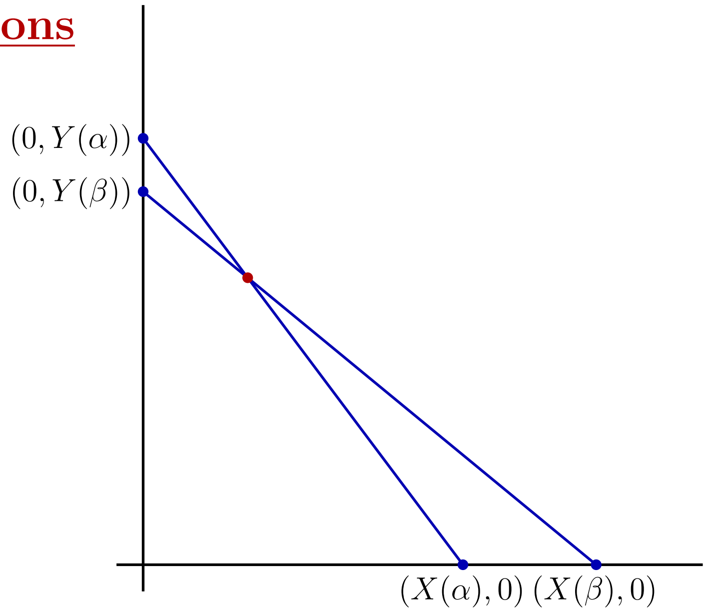
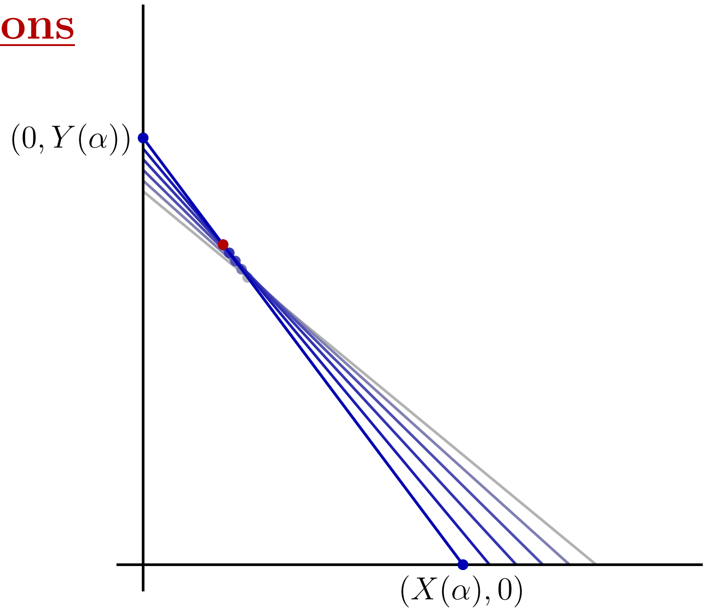


Exercise:

Segments ℓ_α and ℓ_β intersect at the point

$$\left(\frac{X(\alpha)X(\beta)(Y(\beta) - Y(\alpha))}{X(\alpha)Y(\beta) - Y(\alpha)X(\beta)}, \frac{Y(\alpha)Y(\beta)(X(\alpha) - X(\beta))}{X(\alpha)Y(\beta) - Y(\alpha)X(\beta)} \right)$$

Generalization: spacing functions



To find a point on the envelope curve, we need to compute the limit of this intersection point as $\beta \to \alpha$.

That is, we need to find

$$\lim_{\beta \to \alpha} \left(\frac{X(\alpha)X(\beta)(Y(\beta) - Y(\alpha))}{X(\alpha)Y(\beta) - Y(\alpha)X(\beta)}, \frac{Y(\alpha)Y(\beta)(X(\alpha) - X(\beta))}{X(\alpha)Y(\beta) - Y(\alpha)X(\beta)} \right)$$

Some calculus

"Plugging in" α for β gives

$$\left(\frac{X(\alpha)X(\alpha)(Y(\alpha)-Y(\alpha))}{X(\alpha)Y(\alpha)-Y(\alpha)X(\alpha)}, \frac{Y(\alpha)Y(\alpha)(X(\alpha)-X(\alpha))}{X(\alpha)Y(\alpha)-Y(\alpha)X(\alpha)}\right)$$

$$= \left(\frac{0}{0}, \frac{0}{0}\right)$$

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So we try something else ...

$$\lim_{\beta\to\alpha}\left(\frac{X(\alpha)X(\beta)(Y(\beta)-Y(\alpha))}{X(\alpha)Y(\beta)-Y(\alpha)X(\beta)},\frac{Y(\alpha)Y(\beta)(X(\alpha)-X(\beta))}{X(\alpha)Y(\beta)-Y(\alpha)X(\beta)}\right)$$

Some calculus

We get $\lim_{\beta\to\alpha} \frac{X(\alpha)X(\beta)(Y(\beta)-Y(\alpha))}{X(\alpha)Y(\beta)-Y(\alpha)X(\beta)}$

$$= \lim_{\beta\to\alpha} \frac{X(\alpha)X(\beta)(Y(\beta)-Y(\alpha))}{X(\alpha)Y(\beta)-X(\alpha)Y(\alpha)+X(\alpha)Y(\alpha)-Y(\alpha)X(\beta)}$$

$$= \lim_{\beta\to\alpha} \frac{X(\alpha)X(\beta)(Y(\beta)-Y(\alpha))}{X(\alpha)(Y(\beta)-Y(\alpha))-Y(\alpha)(X(\beta)-X(\alpha))}$$

$$= \lim_{\beta\to\alpha} \frac{X(\alpha)X(\beta)(\frac{Y(\beta)-Y(\alpha)}{\beta-\alpha})}{X(\alpha)(\frac{Y(\beta)-Y(\alpha)}{\beta-\alpha})-Y(\alpha)(\frac{X(\beta)-X(\alpha)}{\beta-\alpha})}$$

$$= \frac{X(\alpha)X(\alpha)\cdot\lim_{\beta\to\alpha}\frac{Y(\beta)-Y(\alpha)}{\beta-\alpha}}{X(\alpha)\cdot\lim_{\beta\to\alpha}\frac{Y(\beta)-Y(\alpha)}{\beta-\alpha}-Y(\alpha)\cdot\lim_{\beta\to\alpha}\frac{X(\beta)-X(\alpha)}{\beta-\alpha}}$$

$$= \frac{(X(\alpha))^2Y'(\alpha)}{X(\alpha)Y'(\alpha)-Y(\alpha)X'(\alpha)}$$

Some calculus

Doing the same thing for the y-coordinate, we get

$$\lim_{\beta\to\alpha} \frac{Y(\alpha)Y(\beta)(X(\alpha)-X(\beta))}{X(\alpha)Y(\beta)-Y(\alpha)X(\beta)} = \frac{-(Y(\alpha))^2X'(\alpha)}{X(\alpha)Y'(\alpha)-Y(\alpha)X'(\alpha)}$$

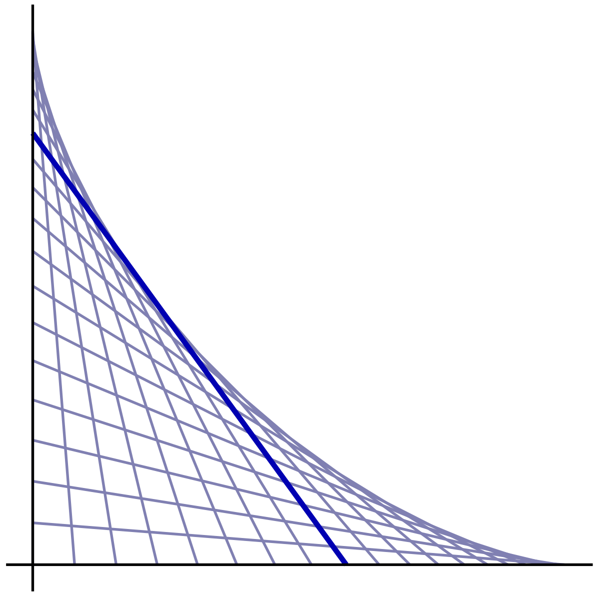
We get the parametrization

$$\left(\frac{(X(\alpha))^2Y'(\alpha)}{X(\alpha)Y'(\alpha)-Y(\alpha)X'(\alpha)}, \frac{-(Y(\alpha))^2X'(\alpha)}{X(\alpha)Y'(\alpha)-Y(\alpha)X'(\alpha)}\right)$$

for the envelope curve.

Example

A ladder of length L slides down a wall. What is the envelope curve?



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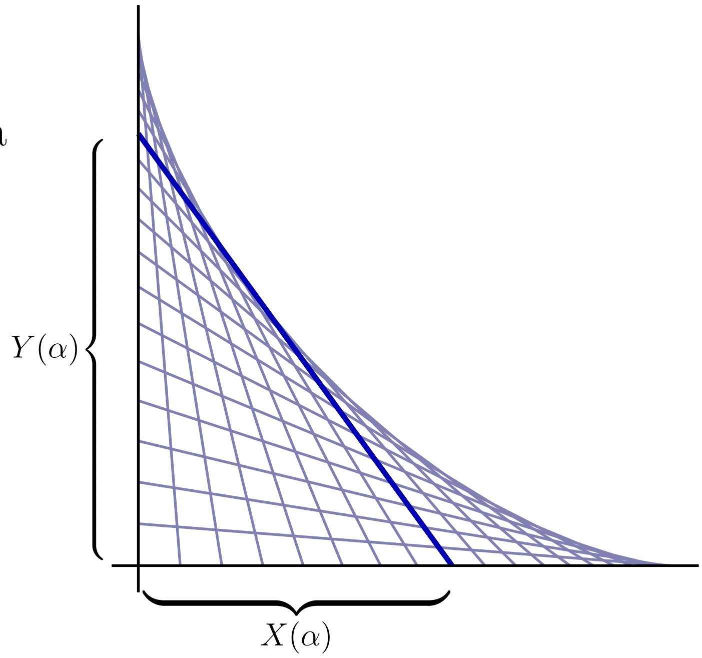

Solution: We want

$$(X(\alpha))^2 + (Y(\alpha))^2 = L^2,$$

so we may as well take

$$X(\alpha) = L\sin(\alpha),$$
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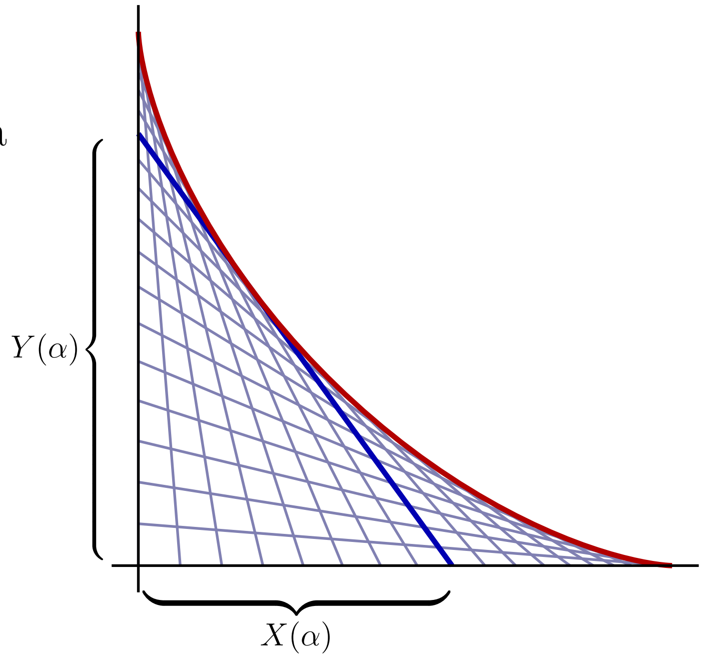

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We get

$$\left(\frac{(X(\alpha))^2 Y'(\alpha)}{X(\alpha)Y'(\alpha) - Y(\alpha)X'(\alpha)}, \frac{-(Y(\alpha))^2 X'(\alpha)}{X(\alpha)Y'(\alpha) - Y(\alpha)X'(\alpha)}\right)$$

$$= (L\sin^3(\alpha), L\cos^3(\alpha))$$

Remarks

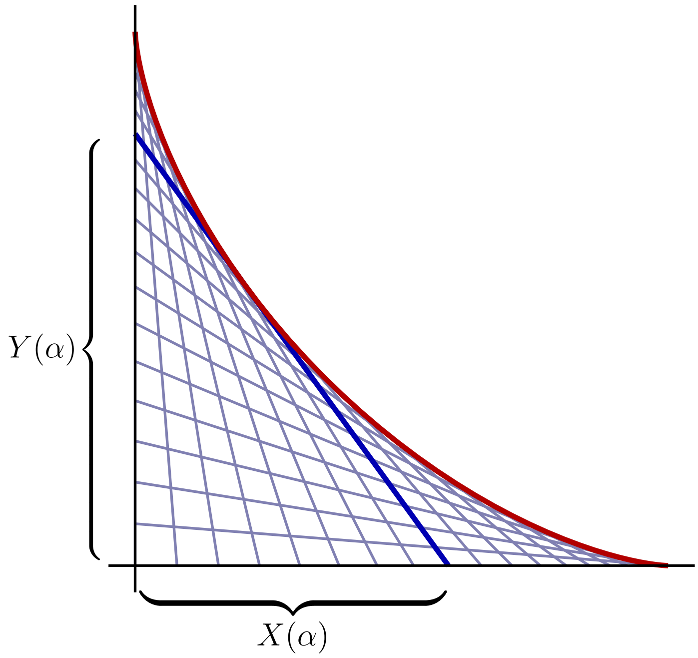
The envelope curve, parametrized by

$$x = L\sin^3(\alpha) \quad \text{and} \quad y = L\cos^3(\alpha)$$

has equation

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = L^{\frac{2}{3}}$$

(This is called an *astroid*.)



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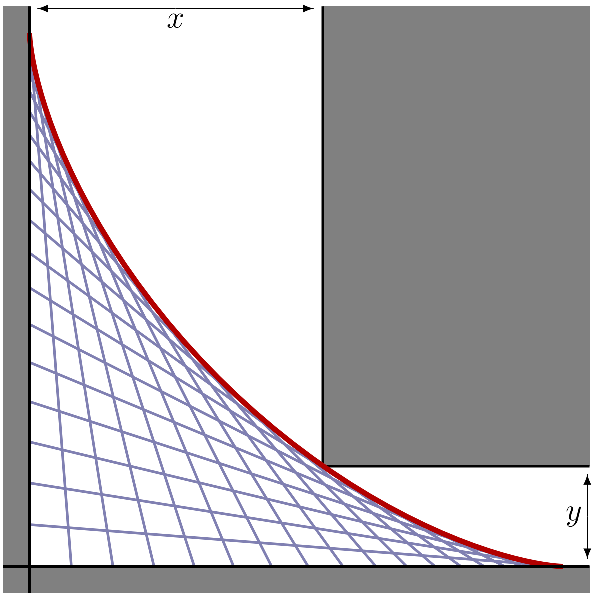

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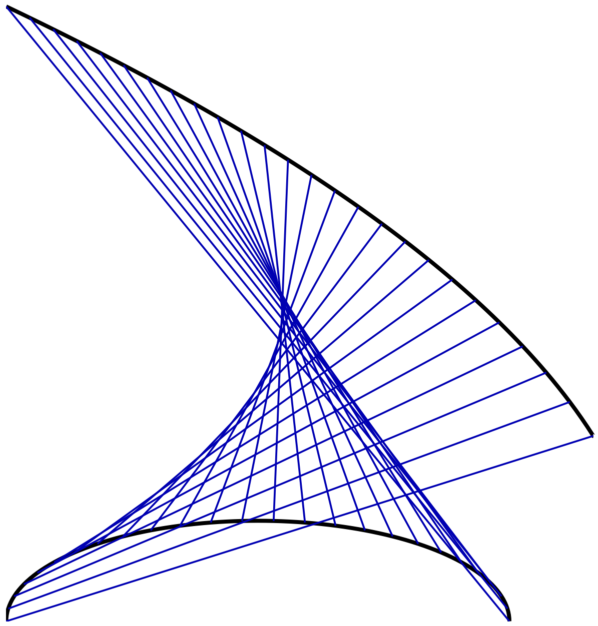
(This is called an *astroid*.)

So if you want to carry your ladder around a corner from a hallway of width x into a hallway of width y, the length of the ladder has to satisfy

$$L^{\frac{2}{3}} \leq x^{\frac{2}{3}} + y^{\frac{2}{3}}$$

Another generalization



Instead of placing the nails along lines, use parametrized curves

$$(X_1(\alpha), Y_1(\alpha)) \text{ and } (X_2(\alpha), Y_2(\alpha))$$

Exercise: Find the intersection point of ℓ_α and ℓ_β, and show that as $\beta \to \alpha$, this point approaches

$$x = \frac{(X_1X_2' - X_1'X_2)(Y_2 - Y_1) - (X_1Y_2' - Y_1'X_2)(X_2 - X_1)}{(X_2' - X_1')(Y_2 - Y_1) - (Y_2' - Y_1')(X_2 - X_1)}$$

$$y = \frac{(Y_1X_2' - X_1'Y_2)(Y_2 - Y_1) - (Y_1Y_2' - Y_1'Y_2)(X_2 - X_1)}{(X_2' - X_1')(Y_2 - Y_1) - (Y_2' - Y_1')(X_2 - X_1)}$$

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