Continued Fractions and Circle Packings

Gregory Quenell

Continued Fractions

A positive number x can be written in the form

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}}$$

where a_0 is a non-negative integer and
a_k is a positive integer for $k \geq 1$.

Notation: We write $[a_0; a_1, a_2, a_3, \ldots]$ for the continued fraction above.

We write $[a_0; a_1, a_2, a_3, \ldots, a_n]$ for a continued fraction that terminates.

Continued Fractions

Example: Write $x_0 = 99\frac{44}{100}$ as a continued fraction.

$$x_0 = 99 + \frac{11}{25}$$

$$= 99 + \frac{1}{(25/11)}$$

Let $a_0 = \lfloor x_0 \rfloor$ and write

$$x_0 = a_0 + r_0$$

with $0 \leq r_0 < 1$.

If $r_0 \neq 0$, let $x_1 = \dfrac{1}{r_0}$, and write

$$x_0 = a_0 + \frac{1}{x_1}$$

Continued Fractions

Example: Write $x_0 = 99\frac{44}{100}$ as a continued fraction.

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$$= 99 + \frac{1}{(25/11)}$$

Let $a_1 = \lfloor x_1 \rfloor$ and write

$$x_1 = a_1 + r_1$$

with $0 \leq r_1 < 1$.

$$= 99 + \cfrac{1}{2 + \cfrac{3}{11}}$$

If $r_1 \neq 0$, let $x_2 = \frac{1}{r_1}$, and rewrite x_1 as

$$a_1 + \frac{1}{x_2}$$

$$= 99 + \cfrac{1}{2 + \cfrac{1}{(11/3)}}$$

Continued Fractions

Example: Write $x_0 = 99\frac{44}{100}$ as a continued fraction.

$$x_0 = 99 + \cfrac{1}{2 + \cfrac{1}{(11/3)}}$$

$$= 99 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{2}{3}}}$$

$$= 99 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{(3/2)}}}$$

Let $a_2 = \lfloor x_2 \rfloor$ and write

$$x_2 = a_2 + r_2$$

with $0 \le r_2 < 1$.

If $r_2 \ne 0$, let $x_3 = \frac{1}{r_2}$, and rewrite x_2 as

$$a_2 + \frac{1}{x_3}$$

Continued Fractions

Example: Write $x_0 = 99\frac{44}{100}$ as a continued fraction.

$$x_0 = 99 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{(3/2)}}}$$

$$= 99 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{2}}}}$$

Let $a_3 = \lfloor x_3 \rfloor$ and write

$$x_3 = a_3 + r_3$$

with $0 \leq r_3 < 1$.

If $r_3 \neq 0$, let $x_4 = \frac{1}{r_3}$, and rewrite x_3 as

$$a_3 + \frac{1}{x_4}$$

Continued Fractions

Example: Write $x_0 = 99\frac{44}{100}$ as a continued fraction.

$$x_0 = 99 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{2}}}}$$

Let $a_4 = \lfloor x_4 \rfloor$ and write

$$x_4 = a_4 + r_4$$

with $0 \leq r_4 < 1$.

This time, $r_4 = 0$, so stop.

$$= 99 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{2+0}}}}$$

$$= [99; 2, 3, 1, 2]$$

Continued Fractions – Useful Facts

- The algorithm terminates – you get $r_k = 0$ for some k – if and only if x_0 is rational.

The number $x = [1; 4, 1, 4, 2]$ is rational ... it's equal to $\frac{64}{53}$	The number $x = [3; 3, 3, 3, 3, \ldots]$ is irrational ... it's equal to $\frac{3+\sqrt{13}}{2}$

- The CFE of a number x is eventually periodic if and only if x is a quadratic surd.

$$\sqrt{7} = [2; 1, 1, 1, 4, 1, 1, 1, 4, 1, 1, 1, 4, \ldots]$$

$$e = [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, \ldots]$$

Continued Fractions – Useful Facts

- Every irrational positive x has a unique continued fraction expansion. Every rational positive x has *two* continued fractions expansions.

$$[2;3,3,1] = 2 + \cfrac{1}{3 + \cfrac{1}{3 + \cfrac{1}{1}}} = 2 + \cfrac{1}{3 + \cfrac{1}{4}} = [2;3,4]$$

If we insist that $[a_0; a_1, a_2, \ldots, a_k, 1]$ always be written as $[a_0; a_1, a_2, \ldots, a_k + 1]$, then every positive x has a unique CFE.

Continued Fractions – Evaluation

- To evaluate a terminating continued fraction, just unwind it from the end:

$$[2;3,4] = 2 + \frac{1}{3+\frac{1}{4}} = 2 + \frac{1}{(13/4)} = 2 + \frac{4}{13} = \frac{30}{13}$$

- For a non-terminating continued fraction, this doesn't work so well:

$$\begin{aligned}&[3;7,15,1,292,1,1,1,2,1,3,1,14,2,\ldots] \\ &\qquad = 3 + \cfrac{1}{7+\frac{1}{15+\frac{1}{1+\frac{1}{292+\frac{1}{1+\cdots}}}}}\end{aligned}$$

Where do you start?

Continued Fractions – Evaluation

Answer: Use Continued Fraction Convergents.

The value of $[3; 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2 \ldots]$ is the limit of the sequence

$$3, \quad [3;7], \quad [3;7,15], \quad [3;7,15,1], \quad [3;7,15,1,292], \quad \ldots$$

That is

$$3, \quad 3+\frac{1}{7}, \quad 3+\frac{1}{7+\frac{1}{15}}, \quad 3+\frac{1}{7+\frac{1}{15+\frac{1}{1}}}, \quad 3+\frac{1}{7+\frac{1}{15+\frac{1}{1+\frac{1}{293}}}}$$

$$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow$$

$$3 \quad \frac{22}{7} \quad \frac{333}{106} \quad \frac{355}{113} \quad \frac{103993}{33102}$$

Continued Fractions – Evaluation

Comments:

- The relatively large coefficient 292 means that the difference between

$$3+\cfrac{1}{7+\cfrac{1}{15+\cfrac{1}{1}}} \quad \text{and} \quad 3+\cfrac{1}{7+\cfrac{1}{15+\cfrac{1}{1+\cfrac{1}{292}}}}$$

  is relatively small.

  Tacking on a large coefficient gives a small change in the value of the continued fraction.

Continued Fractions – Evaluation

Comments:

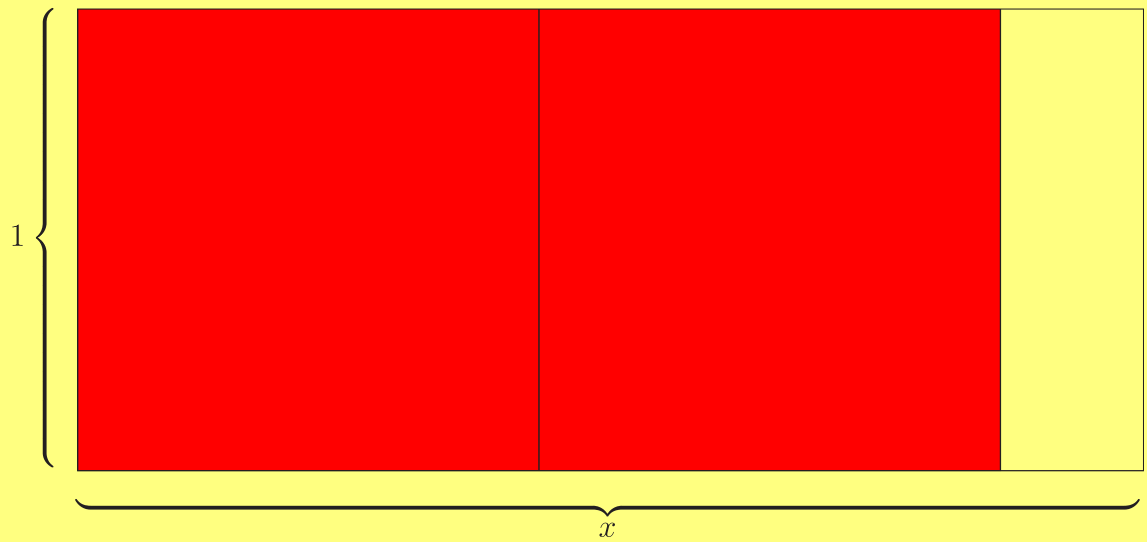
- The continued fraction convergents alternately under- and overestimate the limiting value.

$$
\begin{aligned}
3 &= 3 \\
[3;7] &\approx 3.1428571429 \\
[3;7,15] &\approx 3.1415094340 \\
[3;7,15,1] &\approx 3.1415929204 \\
[3;7,15,1,292] &\approx 3.1415926530 \\
[3;7,15,1,292,1] &\approx 3.1415926539
\end{aligned}
$$

Visualizing Continued Fractions

Given a positive x, draw an x-by-1 rectangle.

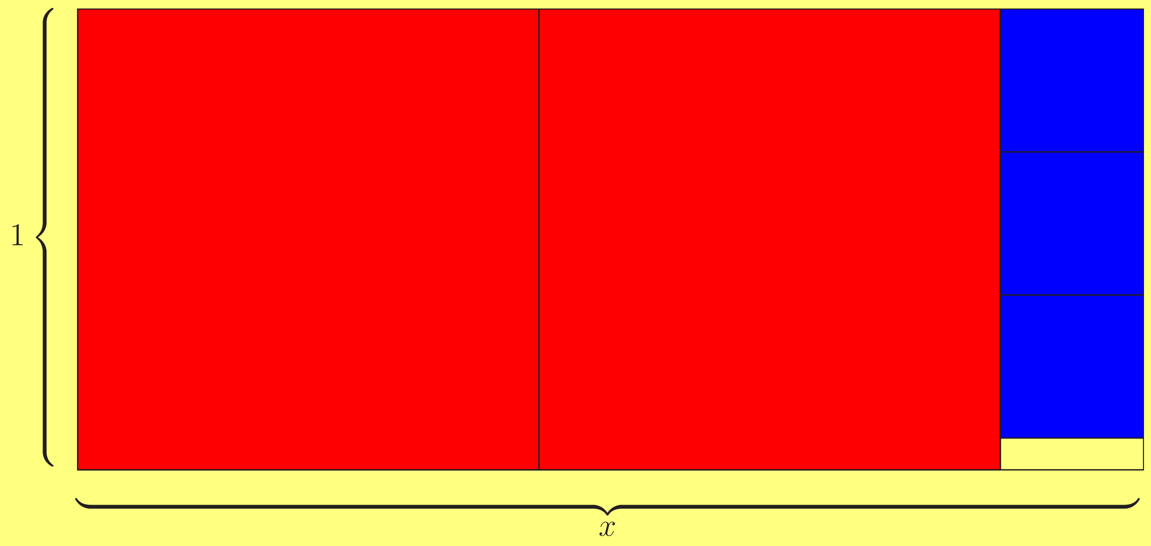
Visualizing Continued Fractions



Starting at the left end, put in as many “horizontal squares” as will fit.

Call this number a_0.

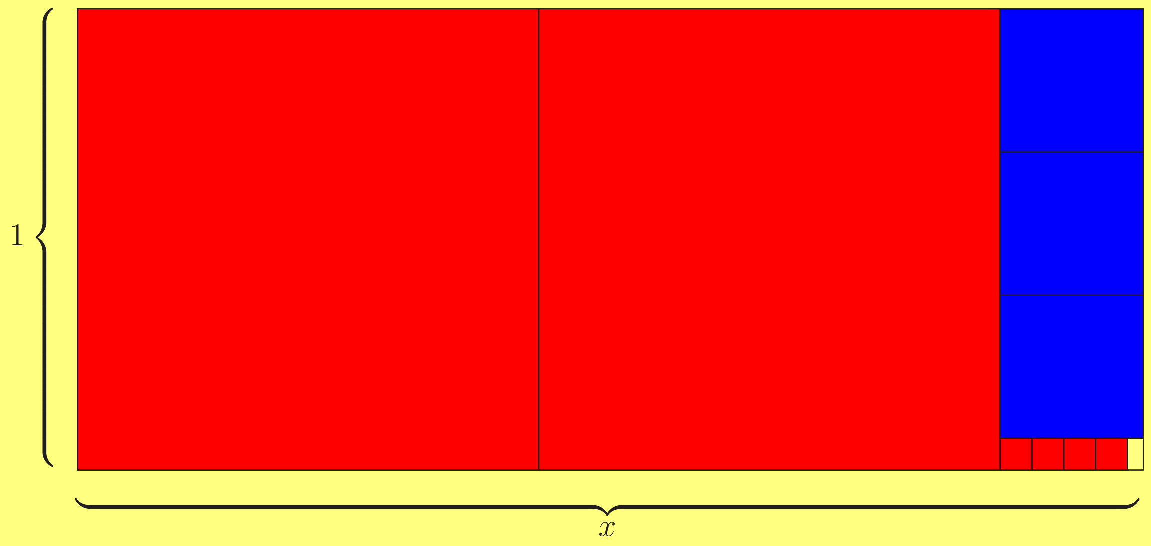
Visualizing Continued Fractions



In the remaining space, put as many "vertical squares" as will fit.

Call this number a_1.

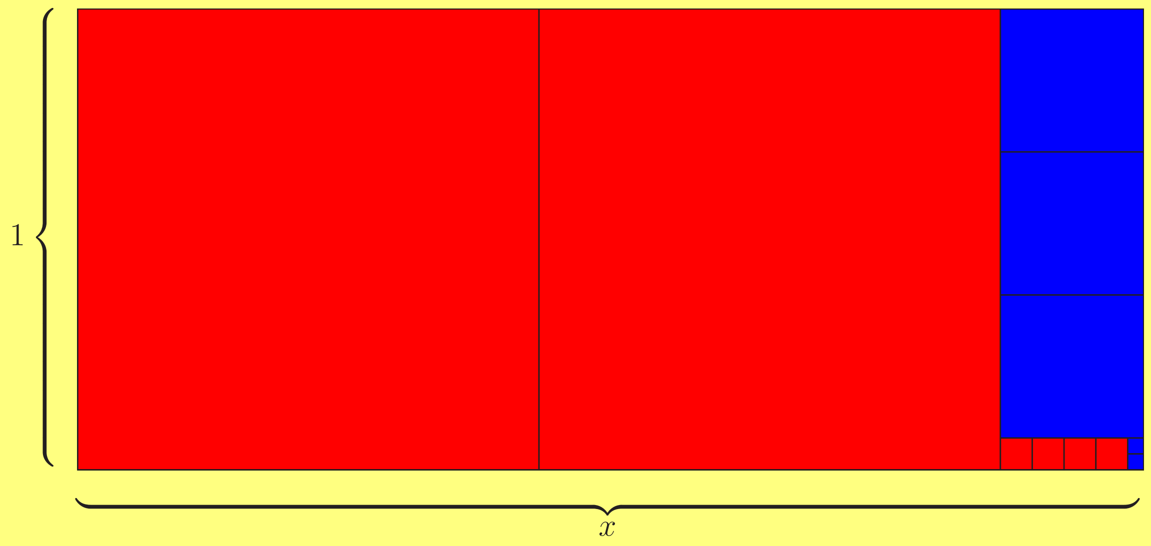
Visualizing Continued Fractions



In the remaining space, put as many horizontal squares as will fit.

Call this number a_2.

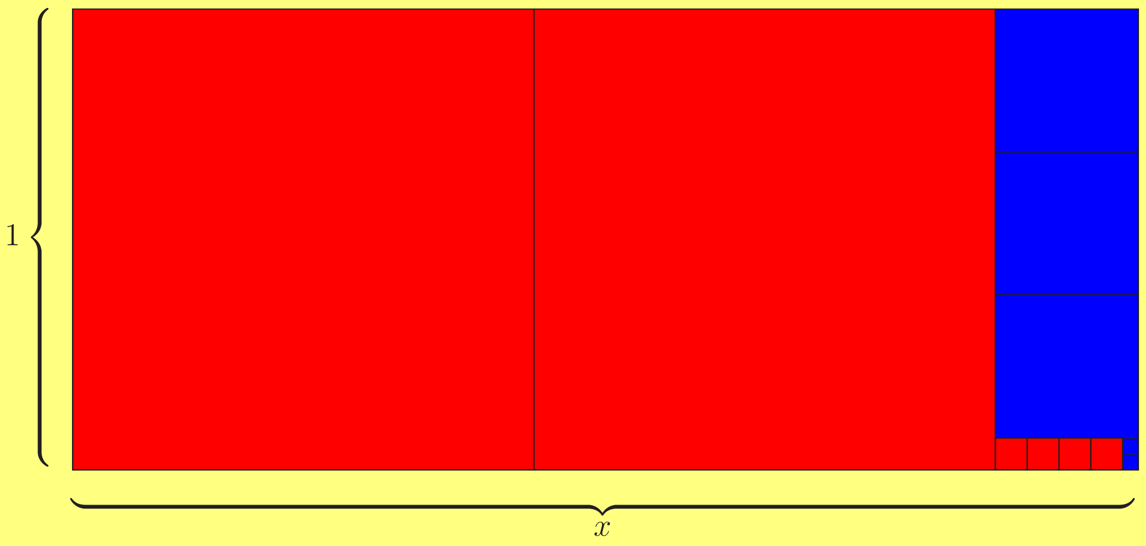
Visualizing Continued Fractions



In the remaining space, put as many vertical squares as will fit.

Call this number a_3.

Visualizing Continued Fractions



Then the "square-packing" sequence we get for x is $\{a_0, a_1, a_2, a_3, \ldots\}$

In this example, the sequence terminates, and we write $\{2, 3, 4, 2\}$.

Visualizing Continued Fractions

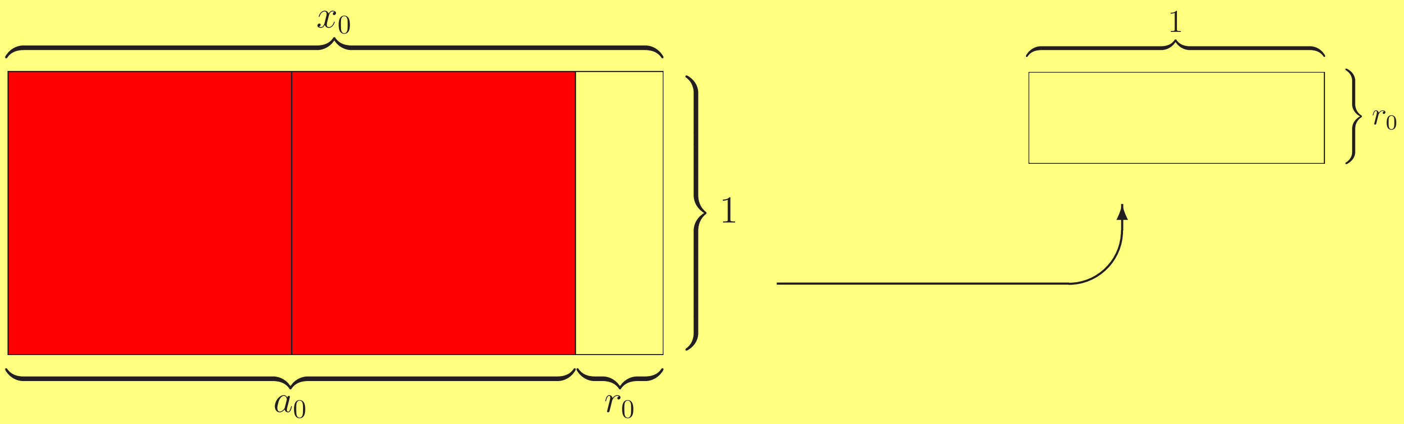
Comment:

This "square-packing" algorithm gives a map

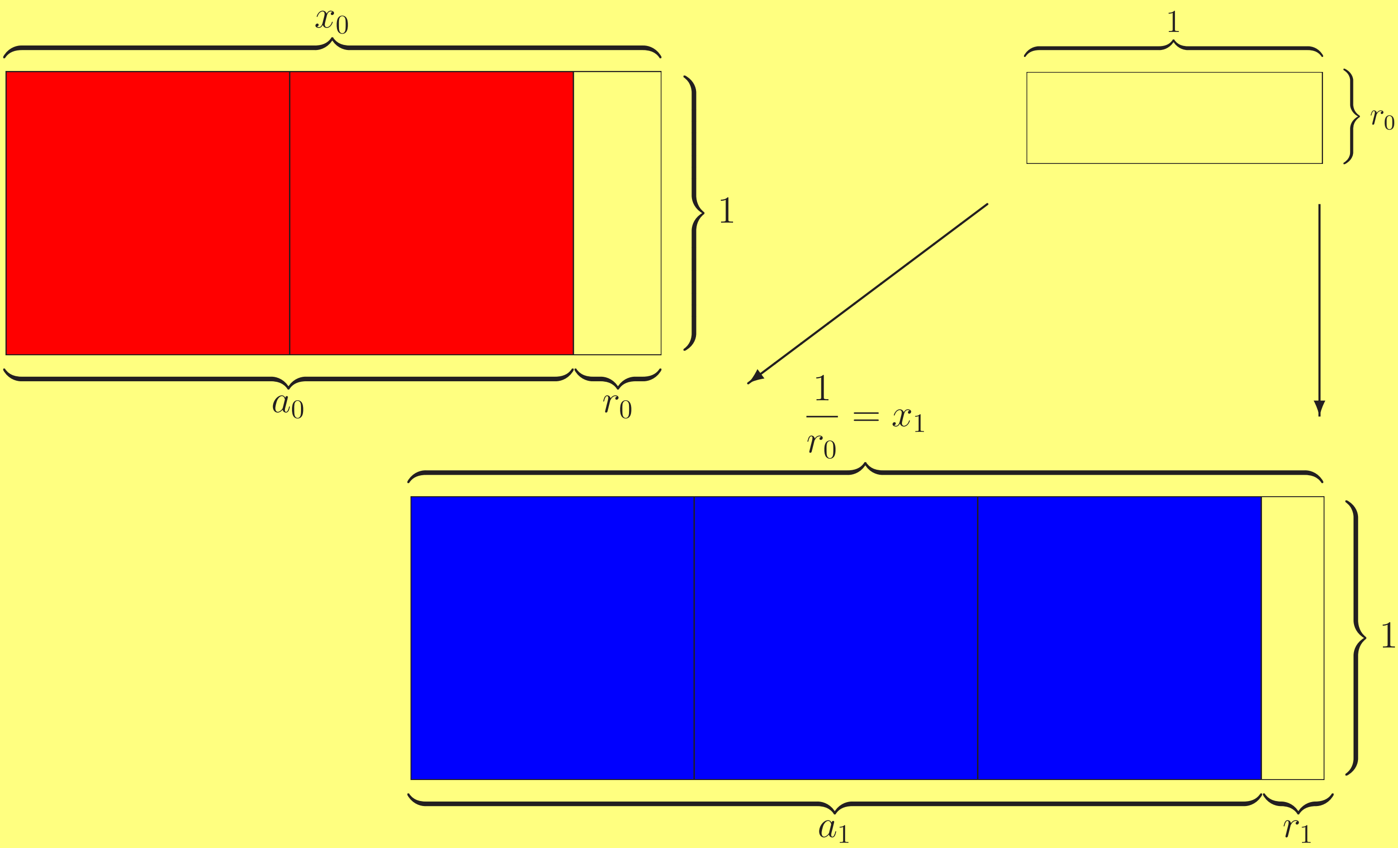
$$\mathcal{S}_{\text{square}} : \mathbb{R}^+ \to \text{sequences of integers}$$

and it's no surprise that $\mathcal{S}_{\text{square}}(x)$ is the continued-fraction expansion of x.

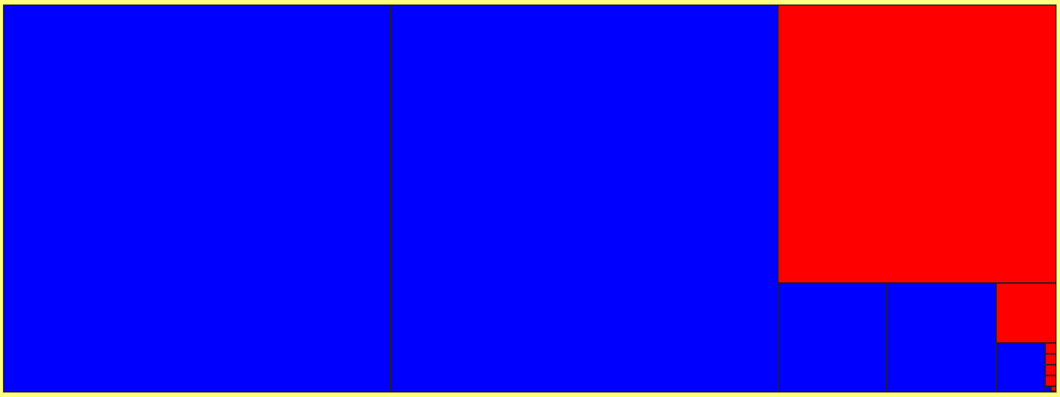
Visualizing Continued Fractions



Visualizing Continued Fractions

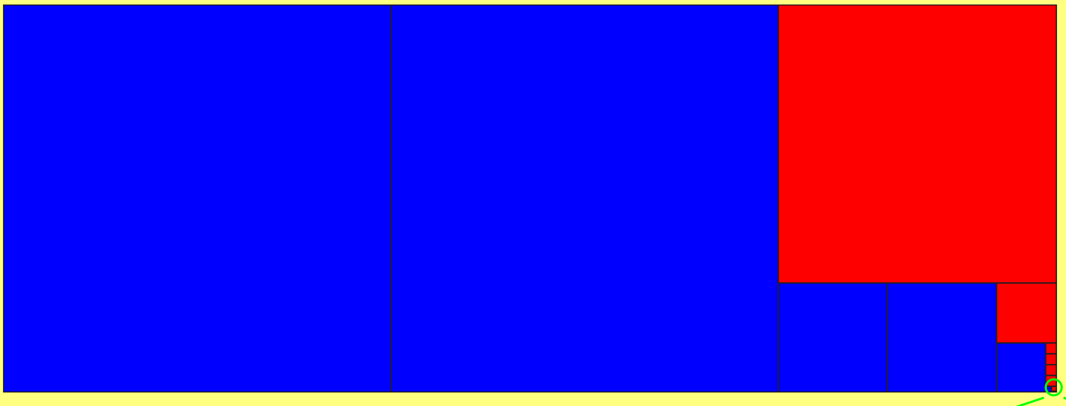


Visualizing Continued Fractions



In a square packing for an irrational number, the horizontal and vertical squares never quite fill up the space.

Visualizing Continued Fractions



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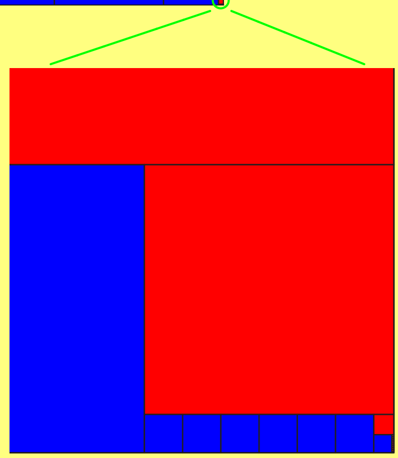
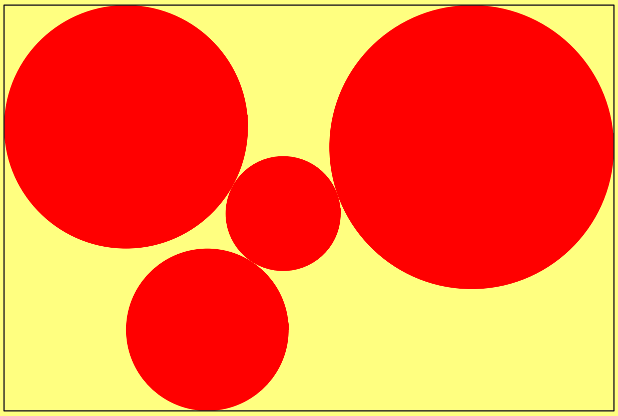
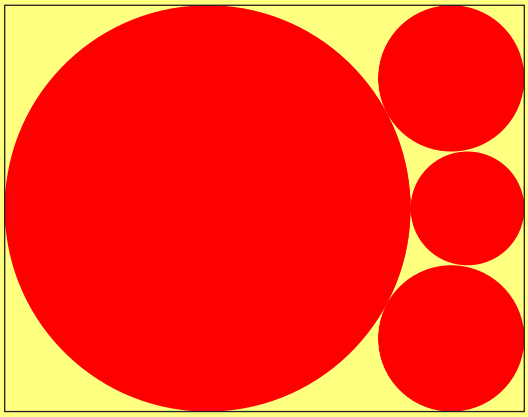


Circle Packings

A *configuration of circles* is an arrangement of circles in which no two circles have overlapping interiors.



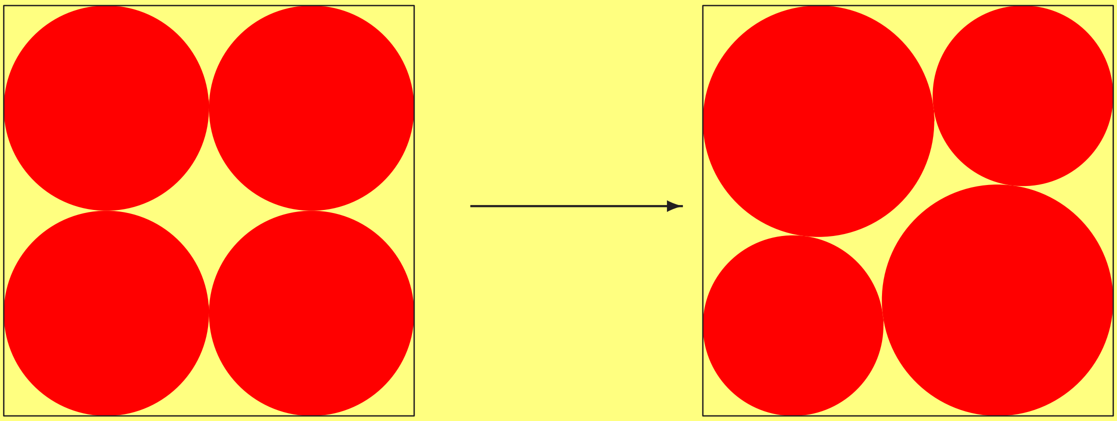
A *circle packing* of a bounded region on the plane or a compact surface is a configuration in which all the interstices are curvilinear triangles.



Circle Packings

A circle packing is special because it is *rigid*:

the packing's geometry is determined by its combinatorics.



This configuration is not rigid. There is a quadrilateral in the middle, and the circles can shift without changing their tangencies.

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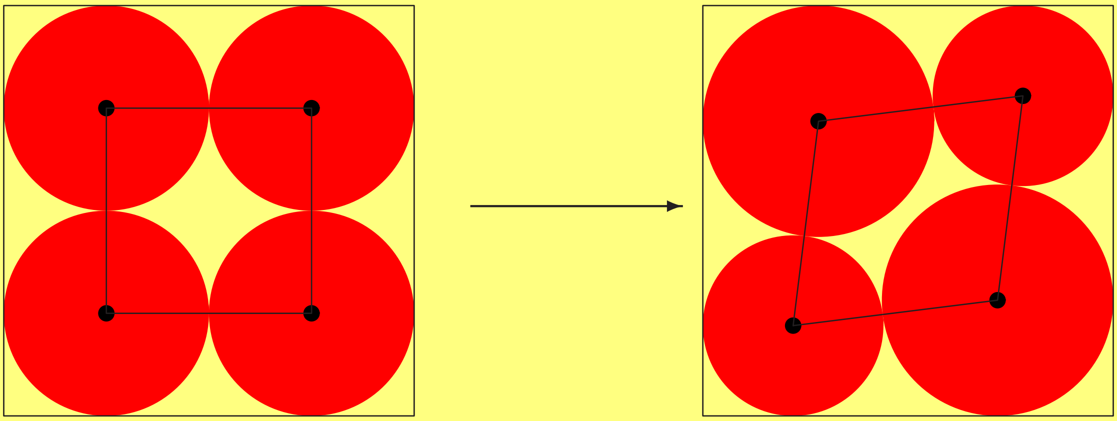

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The quadrilateral shows up clearly in the *tangency graph*.

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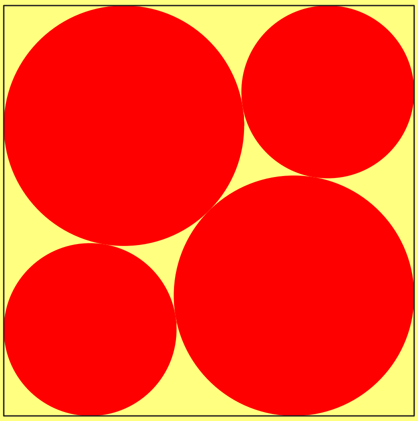

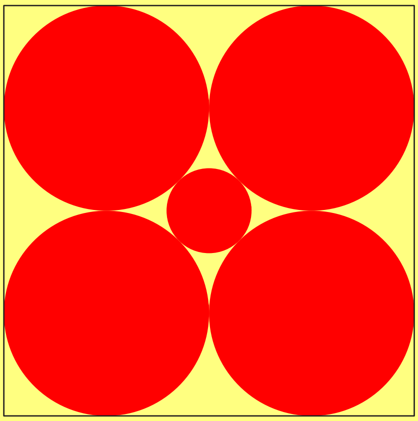

These are circle packings, and they are rigid. All the interstices are curvilinear triangles.

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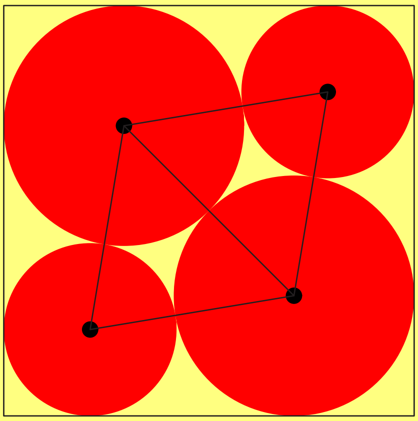
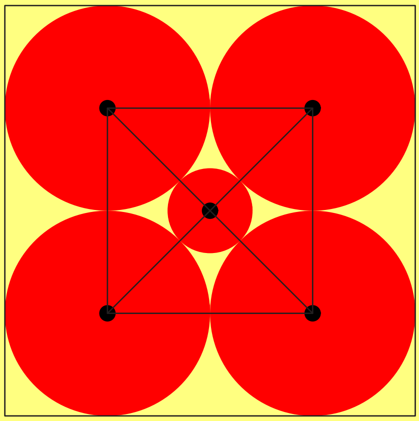




These are circle packings, and they are rigid. All the interstices are curvilinear triangles.

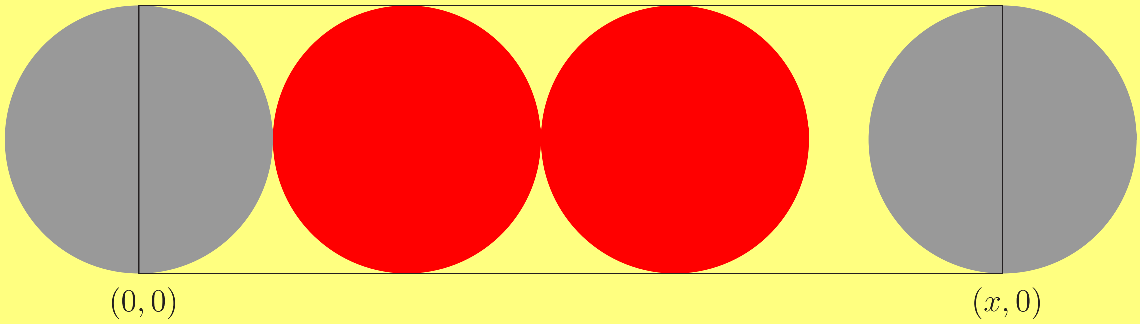
The tangency graph of a packing is always a triangulation.

The Brooks Parameter

Given a positive x, form a curvilinear quadrilateral using reference circles with diameter 1 centered at $(0, 1/2)$ and $(x, 1/2)$.

The Brooks Parameter

$\{2, \ldots\}$



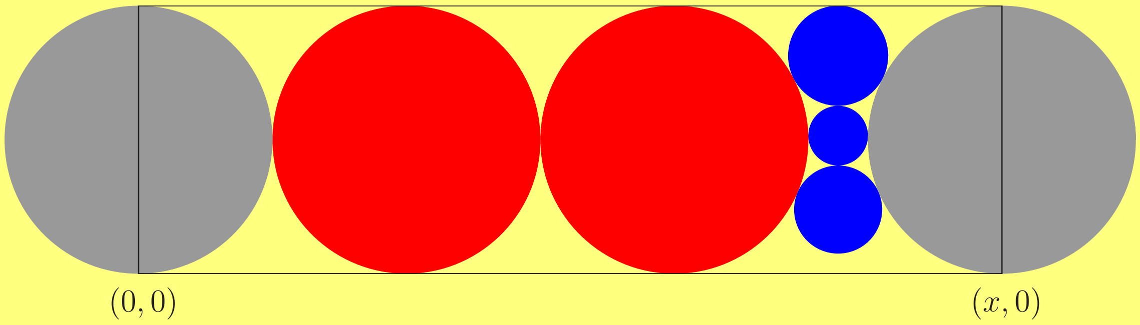
Given a positive x, form a curvilinear quadrilateral using reference circles with diameter 1 centered at $(0, 1/2)$ and $(x, 1/2)$.

Starting at the left end, put in as many "horizontal circles" as you can. A horizontal circle is tangent to the top, bottom, and left sides of its enclosing quadrilateral.

Call this number b_0.

The Brooks Parameter

$\{2, 3, \ldots\}$

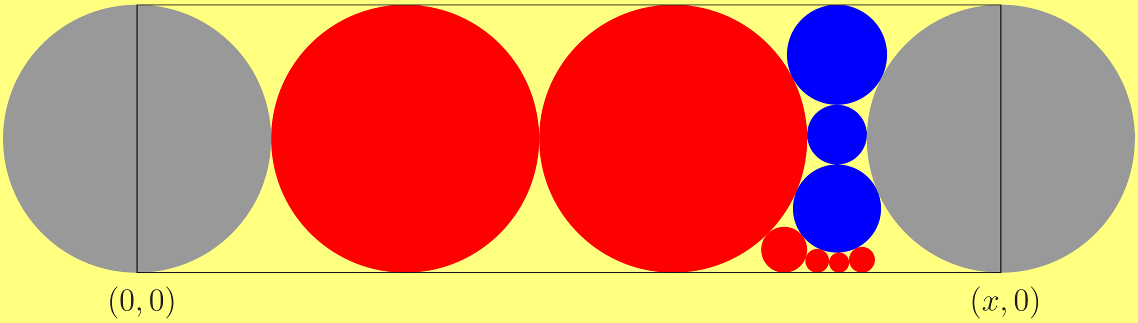


Now start at the top of the remaining unfilled quadrilateral, and put in as many "vertical circles" as you can. A vertical circle is tangent to the top, left, and right sides of its enclosing quadrilateral.

Call this number b_1.

The Brooks Parameter

$\{2, 3, 4, \ldots\}$

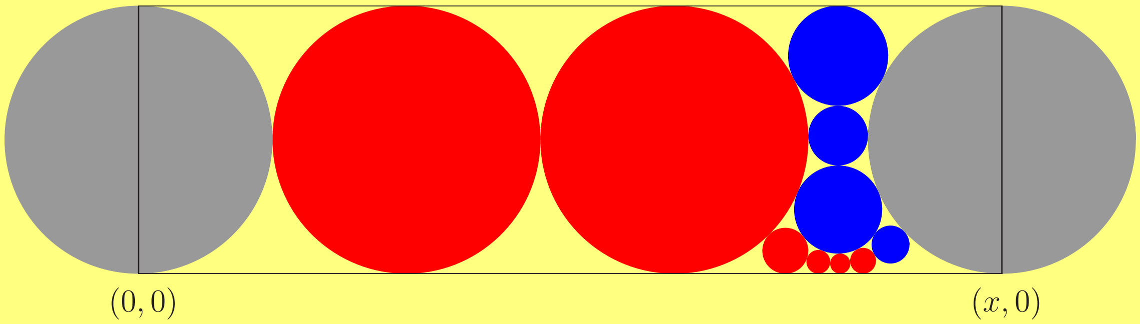


Now put as many horizontal circles as you can into the remaining unfilled quadrilateral, starting at the left end.

Call this number b_2.

The Brooks Parameter

$\{2, 3, 4, 1, \ldots\}$

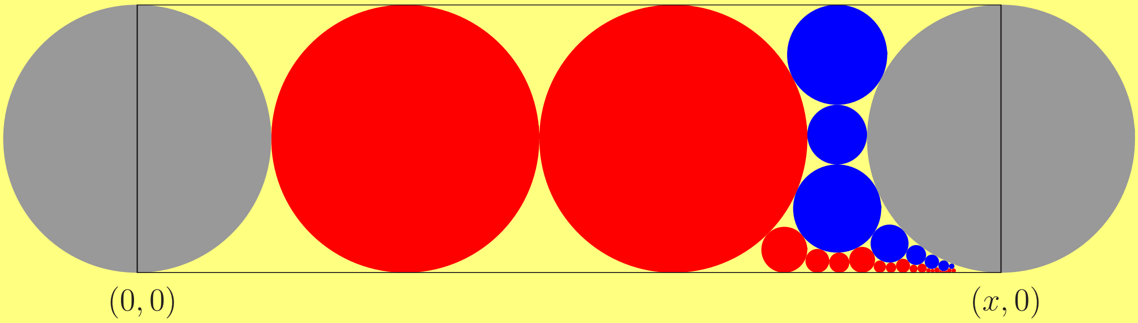


Now start at the top of the remaining unfilled quadrilateral, and put in as many vertical circles as you can.

Call this number b_3.

The Brooks Parameter

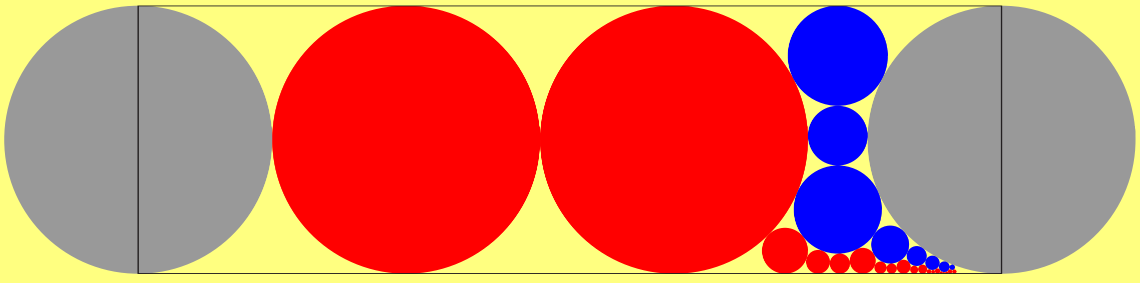
$\{2, 3, 4, 1, 3, 1, 2, 1, 3, 1, 5, 1, \ldots\}$



Continue alternately adding horizontal and vertical circles until either

- the last circle in a row or column is tangent on all four sides, or
- you run out of time or patience.

The Brooks Parameter

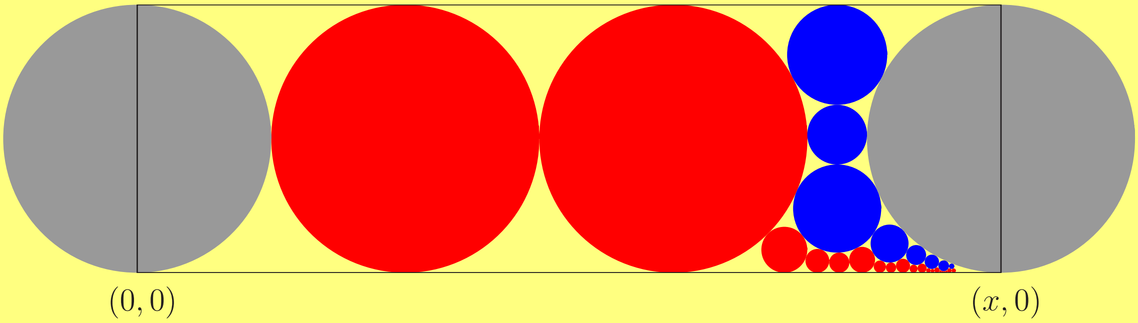


This algorithm gives us a map $\mathcal{S}_{\text{circle}} : [1, \infty) \to$ sequences of integers.

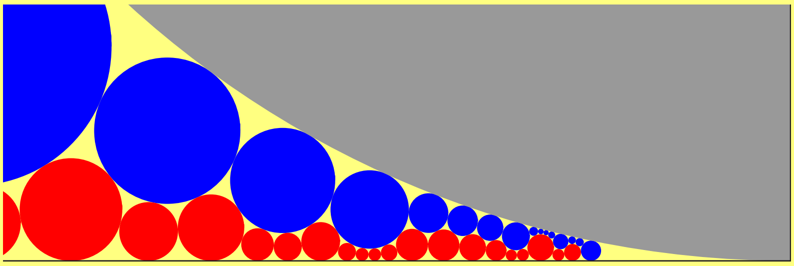
Note that $\mathcal{S}_{\text{circle}}(x)$ is a finite sequence only if the last circle in a row or column is tangent to all four sides of its enclosing quadrilateral.

In this case, we have constructed a *packing* of the original quadrilateral.

The Brooks Parameter

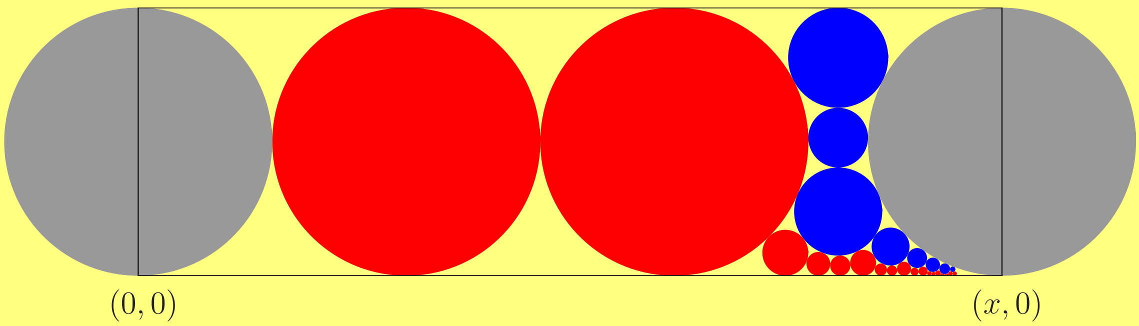


Or the process may just go on forever.



The Brooks Parameter

$\{2, 3, 4, 1, 3, 1, 2, 1, 3, 1, 5, 1, \ldots\}$



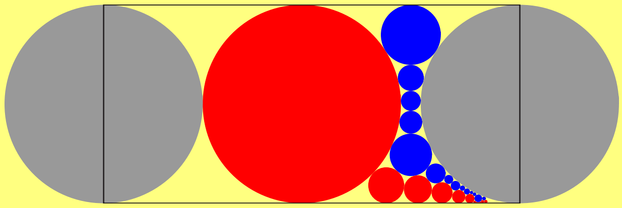
Define the *Brooks parameter* $r_{\text{circle}} : [1, \infty) \to \mathbb{R}^+$

by reading $\mathcal{S}_{\text{circle}}(x)$ as a continued fraction.

For the x in the picture (approximately 3.22), we have

$$r_{\text{circle}}(x) \approx [2; 3, 4, 1, 3, 1, 2, 1, 3, 1, 5, 1, \ldots] \approx 2.312$$

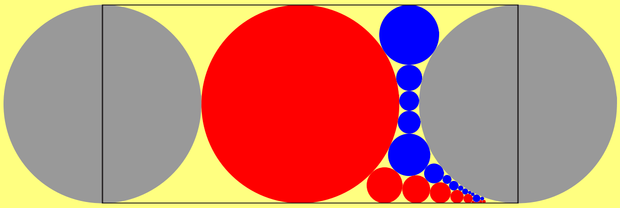
The Brooks Parameter



Observations:

- We have $r_{\text{circle}}(2) = 1$, $r_{\text{circle}}(3) = 2$, and in general, $r_{\text{circle}}(n+1) = n$ if n is an integer.
- The function $r_{\text{circle}}(x) - x$ is 1-periodic.
- If $r_{\text{circle}}(x)$ is rational, then the original x-by-1 curvilinear quadrilateral is packable.

The Brooks Parameter

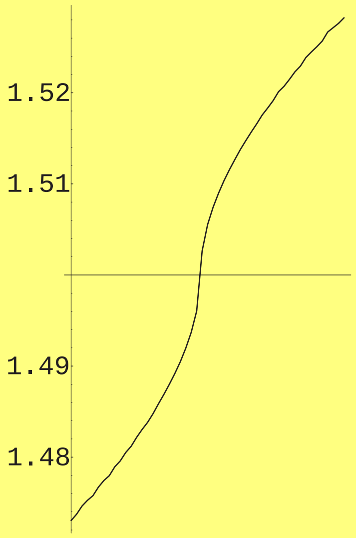


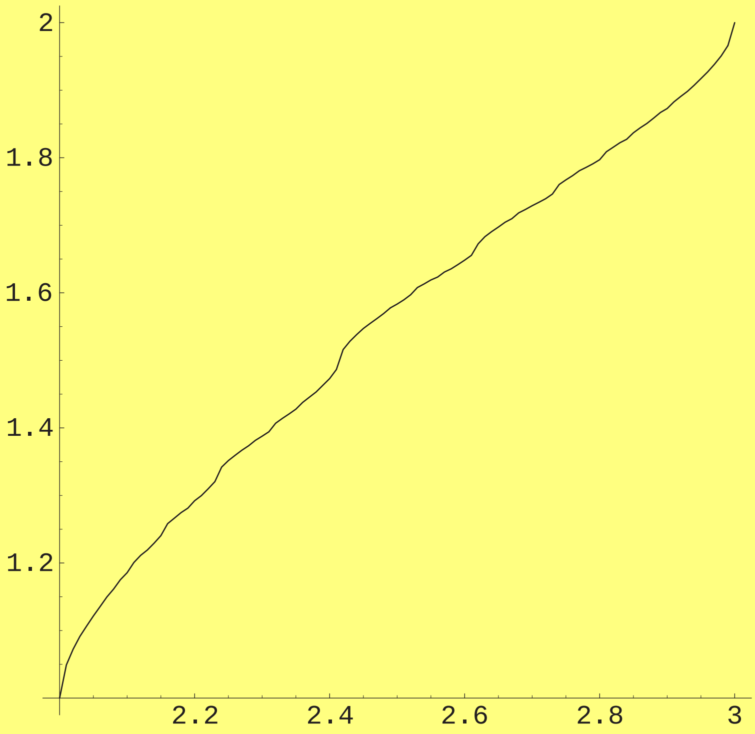
Questions:

- Is $r_{\text{circle}}(x)$ differentiable?
- Is $r_{\text{circle}}(x)$ continuous?
- Is $r_{\text{circle}}(x)$ increasing? How closely does it mimic the analogous function for square packing (namely, $r_{\text{square}}(x) = x$)?
- Is $r_{\text{circle}}(x)$ useful?

The Brooks Parameter

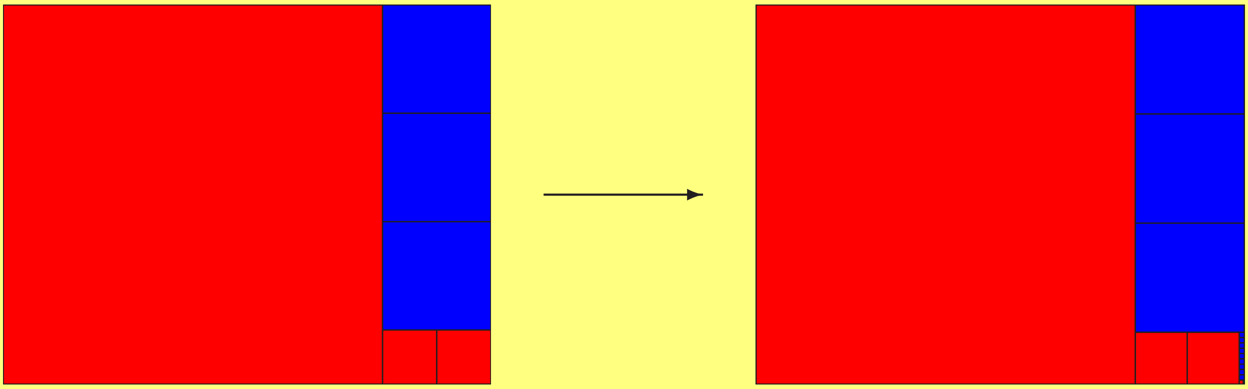
The function $r_{\text{circle}}(\cdot)$ is computable (in theory, at least); here's a graph.





Continuity

Why is $r_{\text{square}}(\cdot)$ continuous?



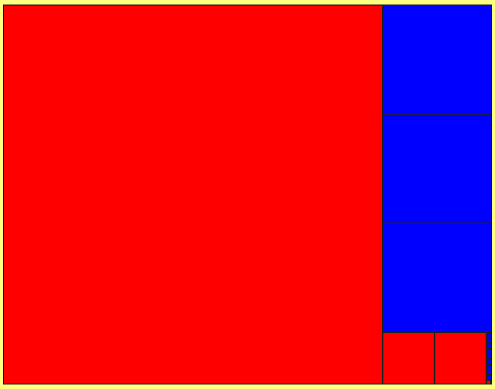
$$r_{\text{square}}(x_0) = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2}} \qquad r_{\text{square}}(x_0+\delta) = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}$$

When we slide from a rational number x_0 to $x_0 + \delta$, we introduce some new coefficients (starting here with a_3). By taking δ sufficiently small, we can make a_3 as large as we want, so that the new term $\frac{1}{a_3 + \cdots}$ can be made arbitrarily small.

Continuity

$$r_{\text{square}}(x_0) = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cfrac{1}{a_5 + \cdots}}}}}$$

$$r_{\text{square}}(x_0 + \delta) = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cfrac{1}{a'_5 + \cdots}}}}}$$

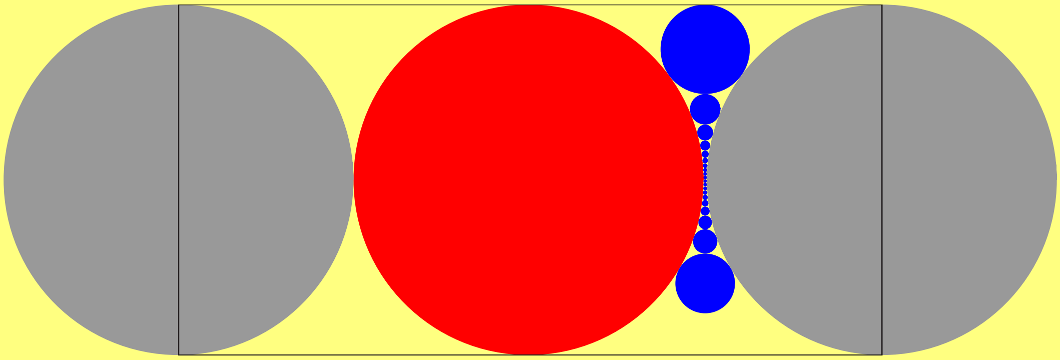


If δ is small enough, then when we slide from an irrational x_0 to $x_0 + \delta$, then the first few coefficients in the CFE do not change.

By choosing δ sufficiently small, we can push the first change in coefficients as far out as we like, and thus make the change in $r_{\text{square}}(x)$ arbitrarily small.

Continuity

The function $r_{\text{circle}}(\cdot)$ is continuous for the same reasons.

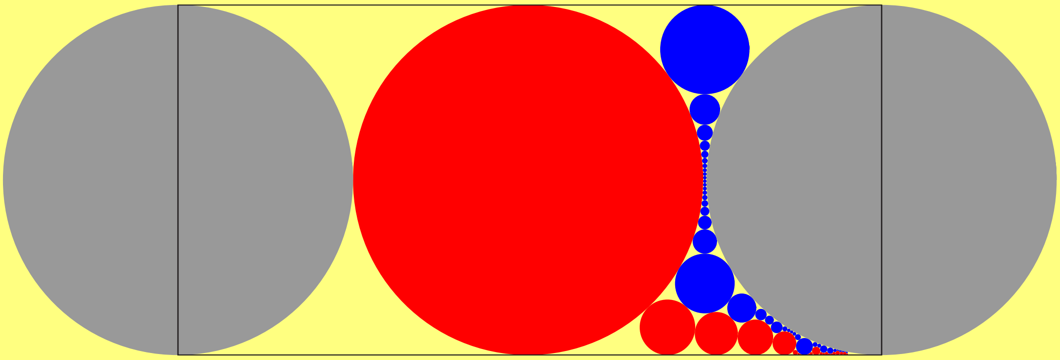


$$r_{\text{circle}}(2+\delta) \;=\; 1 + \frac{1}{b_1}$$

When we introduce a new row or column of circles, we can choose δ so as to make the number of new circles as large as we like.

Continuity

The function $r_{\text{circle}}(\cdot)$ is continuous for the same reasons.

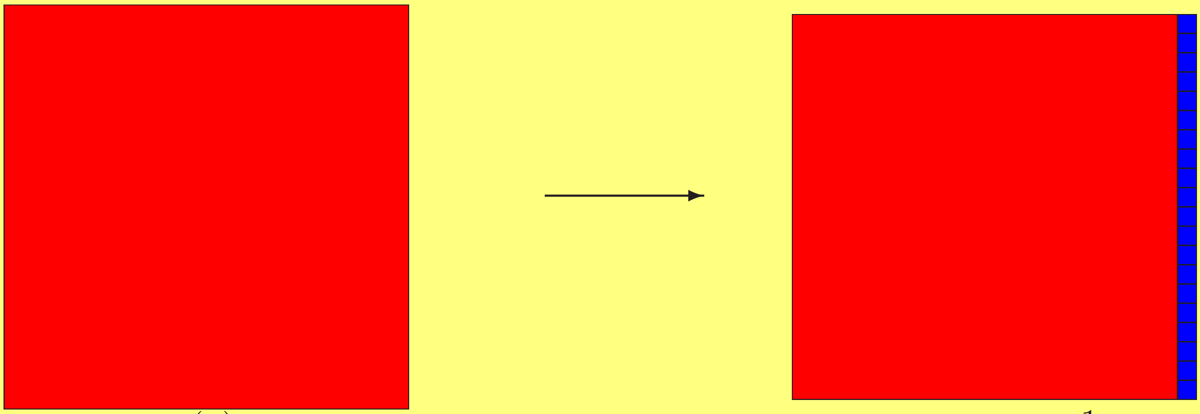


$$r_{\text{circle}}(x_0+\delta) \;=\; 1+\cfrac{1}{b_1+\cfrac{1}{b_2+\cfrac{1}{b_3+\cfrac{1}{b_4+\cdots}}}}$$

And if we start at an irrational x_0, we may make δ small enough so that it does not disturb b_0, b_1, b_2, $\ldots$, b_n for whatever n we choose.

Differentiability

Why is $r_{\text{square}}(\cdot)$ differentiable at 1?



$r_{\text{square}}(1) = 1$

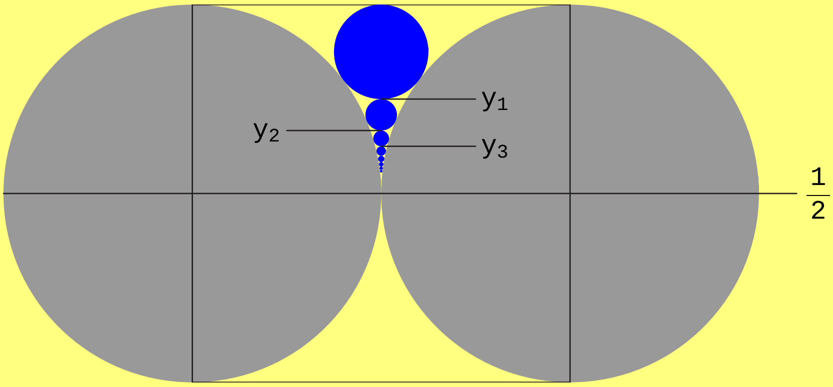
$$r_{\text{square}}(1+\varepsilon) \approx 1 + \frac{1}{(1/\varepsilon)} = 1 + \varepsilon$$

The new column contains approximately $1/\varepsilon$ squares, so $r_{\text{square}}(1+\varepsilon) \approx 1+\varepsilon$, and

$$r'_{\text{square}}(1) = \lim_{\varepsilon\to 0} \frac{(1+\varepsilon)-1}{\varepsilon} = 1$$

Differentiability

What is $r'_{\text{circle}}(1)$?

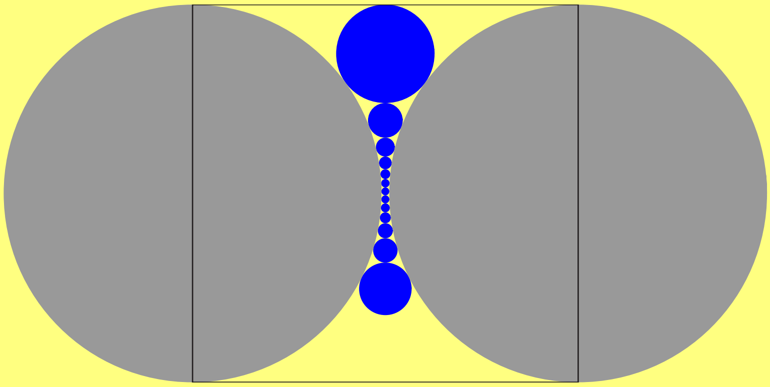


Exercise: Show that $y_k = \frac{1}{2} + \frac{1}{2(k+1)}$ for $k = 1, 2, 3, \ldots$.

Corollary: The diameter of the k^{th} circle from the top is $\frac{1}{2(k^2+k)}$.

Differentiability

Reasoning *very* roughly, it takes on the order of $\dfrac{1}{\sqrt{2\varepsilon}}$ circles to get down to a diameter of ε.



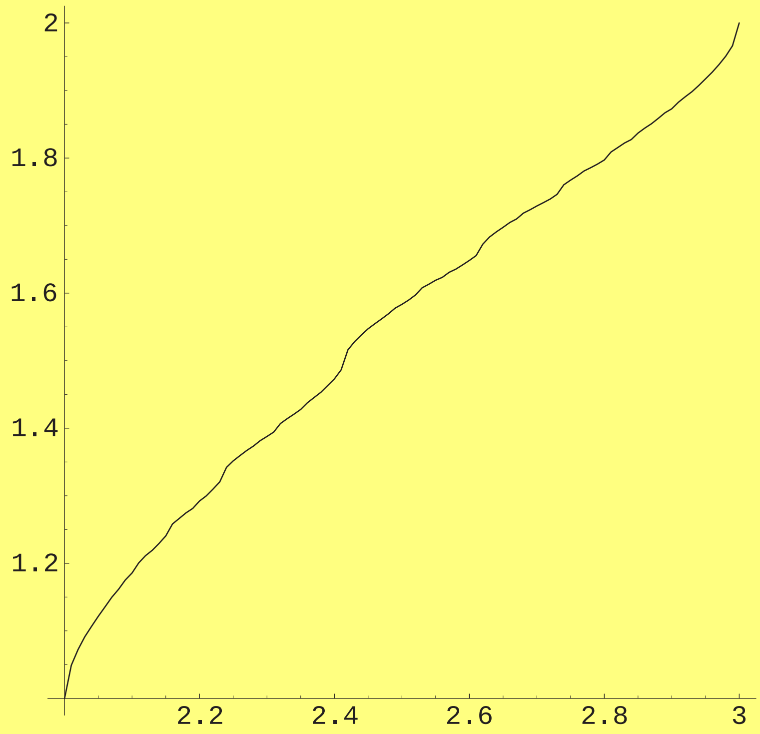
The new column of circles with diameter ε at the middle therefore contains on the order of $\dfrac{2}{\sqrt{2\varepsilon}}$ circles. We get $r_{\text{circle}}(1+\varepsilon) \approx 0 + \dfrac{1}{(2/\sqrt{2\varepsilon})}$, so that

$$r'_{\text{circle}}(1) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \cdot \frac{\sqrt{2\varepsilon}}{2} \to \infty$$

Differentiability

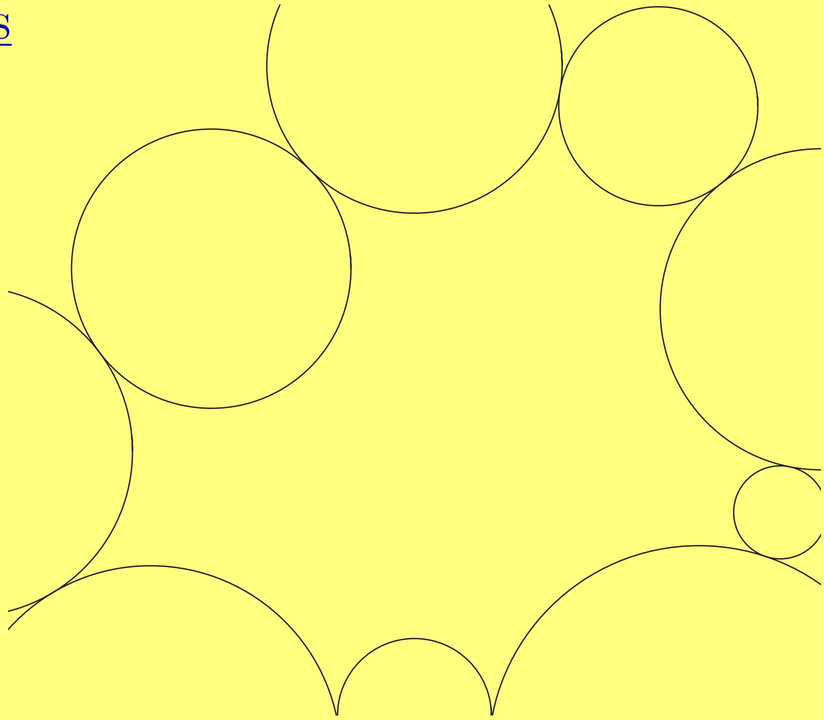
One uses a linear fractional transformation to move any row or column of "new" circles into this position and thus proves the

Theorem (Brooks, 1990): The derivative of r_{circle} is infinite at any x such that $r_{\text{circle}}(x)$ is rational.



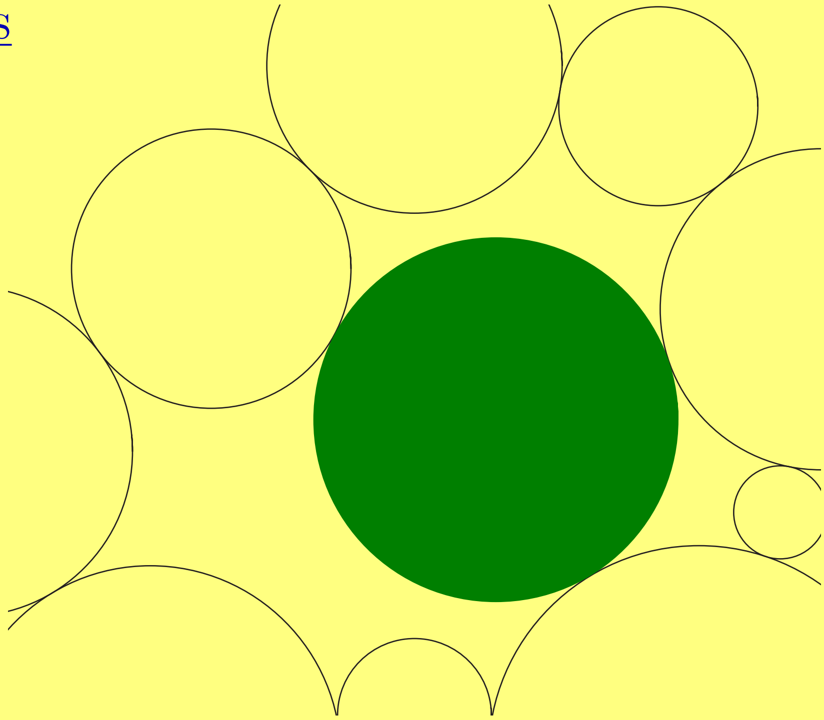
So r_{circle} is an example of a function that is continuous on $[1, \infty)$ but is non-differentiable at a dense set of points.

Applications



Given a region bounded by circular arcs, you can add circles until the regions that remain are all triangles or quadrilaterals.

Applications



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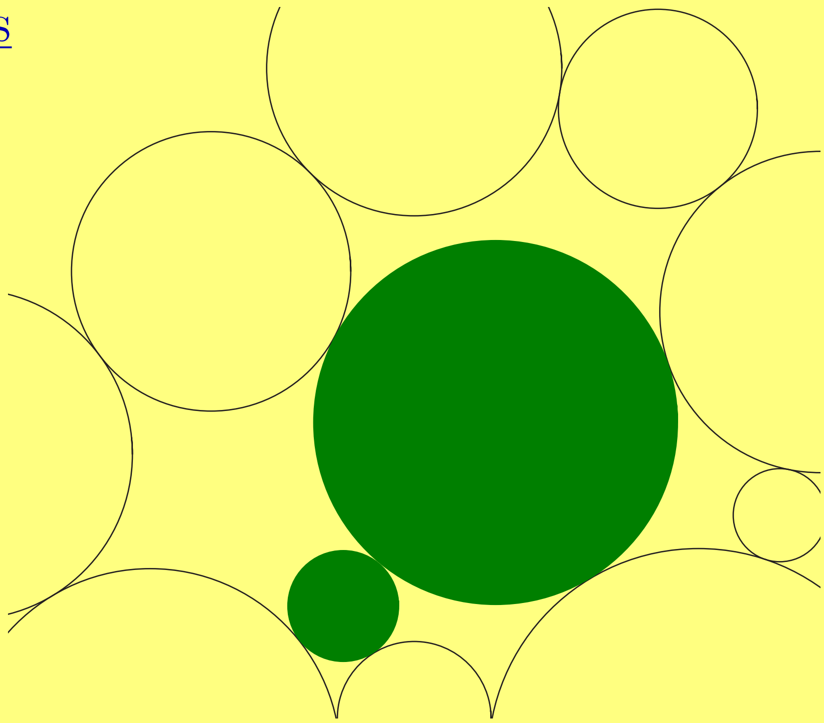
Applications



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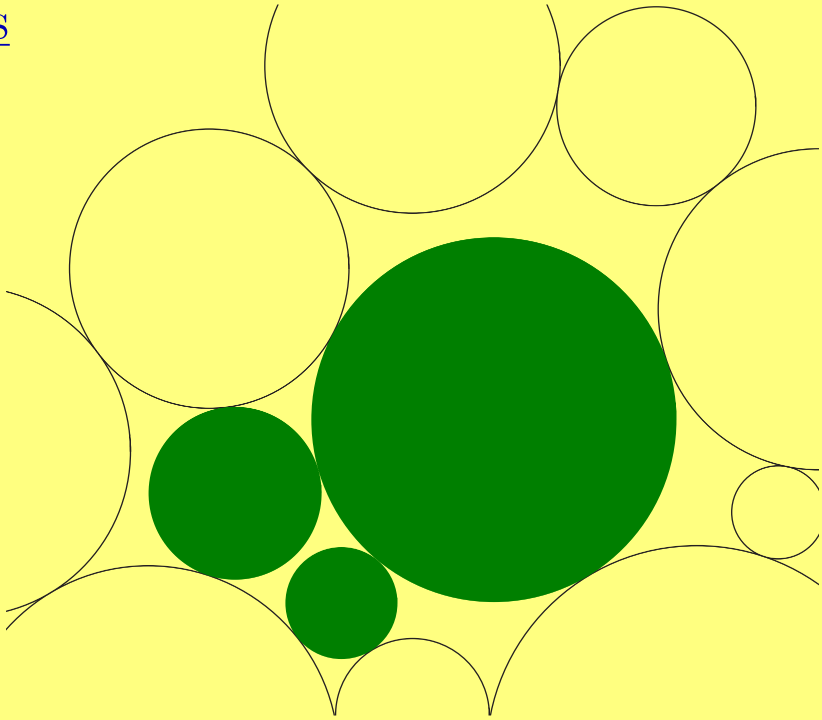

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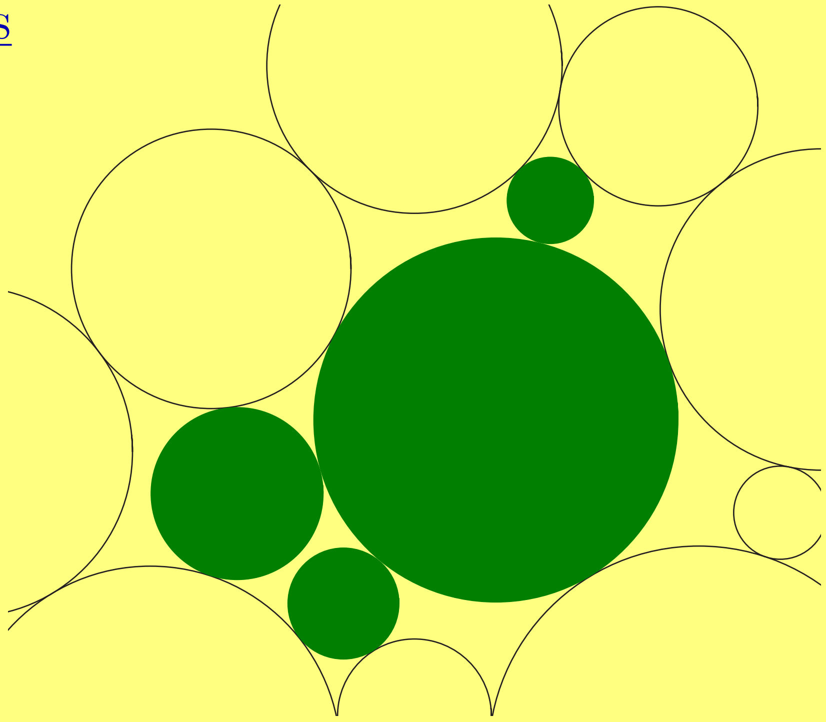
Applications



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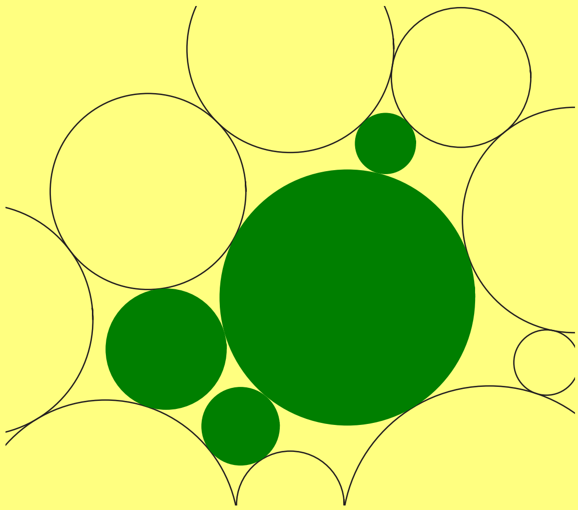
Applications

This can be completed to give a packing if we can find a packing of each of the quadrilaterals.



Sometimes a quadrilateral isn't packable. In that case, its Brooks parameter is irrational. By the continuity of r_{circle}, you can make the Brooks parameter rational by making an arbitrarily small change to the quadrilateral.

So,

Theorem family: Even if you're given a non-packable region, there's always a packable one right nearby.

References

- Robert Brooks, On the deformation theory of classical Schottky groups, *Duke Mathematical Journal* 52, 1985.
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