On the trace of a product

Gregory Quenell

January 1998

We consider the well-known theorem

Theorem 1 *If A and B are square matrices of the same size then* $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$.

This result appears as an exercise in many introductory textbooks on linear algebra ([1, p. 76], [4, p. 216], [6, p. 105], and [7, p. 41], for example). Its placement usually suggests that the expected solution is a nuts-and-bolts proof whose key step involves a change in the order of a sum of products of matrix elements. (See [2, p. 207].) By considering Theorem 1 as a statement about graphs, we develop a more intuitive way to view this result.

We also gain some insight into the way this result extends to the product of more than two matrices. Repeated application of Theorem 1 shows that the trace of a product of more than two matrices in invariant under cyclic permutations of the factors. It is not difficult to construct examples showing that the trace fails to be invariant under other permutations of the factors. A graph-theoretic approach makes it clear why this should be so.

We begin with a finite directed graph G. Let A be an adjacency matrix for G. That is, A is a square matrix with rows and columns indexed by the vertices of G in which the $(i, j)^{\text{th}}$ entry is equal to 1 if there is an edge from vertex i to vertex j and 0 otherwise.

An *n-walk* from vertex i to vertex j in G is a sequence $e_1, e_2, \ldots, e_n$ of n edges such that the initial vertex of e_1 is i, the terminal vertex of e_n is j, and for $k = 1, \ldots, n-1$, the terminal vertex of e_k is the same as the initial vertex of e_{k+1}. A familiar proof by induction (see [3] or [5], among others) establishes the following.

Theorem 2 *For each integer $n \geq 0$, the $(i,j)^{\text{th}}$ entry of the matrix A^n is equal to the number of n-walks in G beginning at vertex i and ending at vertex j.*

An n-walk is *closed* if it begins and ends at the same vertex. It follows from Theorem 2 that the $(i,i)^{\text{th}}$ entry of A^n is equal to the number of closed n-walks at vertex i.

Now suppose that $G_1, G_2, \ldots, G_d$ are directed graphs, each with the same (finite) number of vertices. Using a set of pens with colors $1, 2, \ldots, d$, we draw these d directed graphs, using color k for graph G_k, *on the same set of vertices.* The result is a graph G (actually, a directed multigraph) on the common set of vertices in which each directed edge has some color. We will say that an n-walk $e_1, e_2, \ldots, e_n$ in G has *color sequence* $c_1, c_2, \ldots, c_n$ if edge e_k has color c_k for $k = 1, \ldots, n$.

Let $A_1, A_2, \ldots, A_d$ be the adjacency matrices for the graphs $G_1, G_2, \ldots, G_d$. A slight variation on the usual proof of Theorem 2 (which we leave to the reader) yields the following.

Theorem 3 *For each sequence $c_1, c_2, \ldots, c_n$ where each c_k is an integer with $1 \leq c_k \leq d$, the $(i,j)^{\text{th}}$ entry of the matrix product $A_{c_1} A_{c_2} \cdots A_{c_n}$ is equal to the number of n-walks beginning at vertex i, ending at vertex j, and having color sequence $c_1, c_2, \ldots, c_n$.*

The diagonal entries of this matrix product count closed walks, so we have the corollary

Corollary 4 *The trace $\mathrm{Tr}(A_{c_1} A_{c_2} \cdots A_{c_n})$ is equal to the total number of closed n-walks in G with color sequence $c_1, c_2, \ldots, c_n$.*

Now a closed n-walk is a cyclic affair, and may be considered to begin and end at any of its vertices. Put another way, for each integer s, there is a one-to-one correspondence between closed n-walks whose color sequence is $c_1, c_2, \ldots, c_n$ and those whose color sequence is $c_{1+s}, c_{2+s}, \ldots c_{n+s}$, where the subscripts are read modulo n. This has the following interpretation in terms of traces:

Theorem 5 *If $A_1, A_2, \ldots, A_d$ are square zero-one matrices all of the same size and $c_1, c_2, \ldots, c_n$ is a sequence in which each c_k is an integer with $1 \leq c_k \leq d$, then for each integer s, we have*

$$\mathrm{Tr}(A_{c_1} A_{c_2} \cdots A_{c_n}) \;\; = \;\; \mathrm{Tr}(A_{c_{1+s}} A_{c_{2+s}} \cdots A_{c_{n+s}})$$

where the subscripts are read modulo n.

When $d = 2$, we get a special case of Theorem 1. As we mentioned above, Theorem 5 is implied by Theorem 1, so we have nothing new here except perhaps some insight into why Theorem 1 works.

A directed graph G may be viewed as an undirected graph if, whenever it has an edge leading from vertex i to vertex j, it also has an edge leading from vertex j back to vertex i. This condition is equivalent to the adjacency matrix A being symmetric. If each of the graphs $G_1, G_2, \ldots, G_d$ is undirected, and we use our colored pens to draw them on a common set of vertices as above, then every n-walk in the resulting colored graph is reversible. More precisely, there is a one-to-one correspondence between closed walks with the color sequence $c_1, c_2, \ldots, c_n$ and closed walks with the color sequence $c_n, c_{n-1}, \ldots, c_1$. This has the following consequence for traces of products.

Theorem 6 *If $A_1, A_2, \ldots, A_d$ are symmetric zero-one matrices all of the same size and $c_1, c_2, \ldots, c_n$ is a sequence in which each c_k is an integer with $1 \leq c_k \leq d$, then*

$$\mathrm{Tr}(A_{c_1} A_{c_2} \cdots A_{c_n}) = \mathrm{Tr}(A_{c_{\sigma(1)}} A_{c_{\sigma(2)}} \cdots A_{c_{\sigma(n)}})$$

for each dihedral permutation σ on n letters.

To get versions of Theorem 5 and Theorem 6 in which the matrices may have entries other than zero and one while maintaining our graph-theoretic approach, we introduce the idea of a weighted graph. This is simply a graph in which each edge has a weight assigned to it. For our purposes, this weight may be a real or complex number (or, for that matter, may come from any commutative ring). The $(i, j)^{\text{th}}$ entry of an adjacency matrix for a weighted graph is equal to the weight of the directed edge from vertex i to vertex j. Given an n-walk $e_1, e_2, \ldots, e_n$ in a weighted graph, we will say that the *weight* of the n-walk is equal to the product of the weights of the edges it traverses.

With these definitions, it is straightforward to prove the generalization of Theorem 2

Theorem 7 *If A is an adjacency matrix for a weighted directed graph G, then the $(i, j)^{\text{th}}$ entry of A^n is equal to the sum of the weights of all n-walks from vertex i to vertex j in G.*

Any matrix M may then be viewed as the adjacency matrix of a weighted, directed graph in which each vertex i is the beginning point for directed edges going to every vertex of the graph (including i itself) and the weight assigned to the edge going from vertex i to vertex j is the $(i, j)^{\text{th}}$ entry of the matrix M.

Extending this idea, given a collection $M_1, M_2, \ldots, M_d$ of square matrices of the same size along with our d different colored pens, we may construct a weighted, colored graph G. This graph is quite messy to draw, because of its large number of edges. If G has m vertices, then each vertex of G is the beginning point for m weighted edges of each of the d colors. Still, matrix multiplication effectively counts walks in the graph G, and the usual proof of Theorem 2 may be adapted to weighted graphs with colored edges to show

Theorem 8 *Let $M_1, M_2, \ldots, M_d$ be square matrices all of the same size, and form the weighted colored graph G as above. Given a sequence $c_1, c_2, \ldots, c_n$ in which each c_k is an integer with $1 \leq c_k \leq d$, the $(i, j)^{\text{th}}$ entry of the product*

$$M_{c_1} M_{c_2} \ldots M_{c_n}$$

is equal to the sum of the weights of all the n-walks from vertex i to vertex j in G which have color sequence $c_1, c_2, \ldots, c_n$.

We then have

Corollary 9 *In the situation in Theorem 8, the trace $\mathrm{Tr}(M_{c_1} M_{c_2} \cdots M_{c_n})$ is equal to the sum of the weights of all the closed n-walks in the graph G with color sequence $c_1, c_2, \ldots, c_n$.*

Next we observe as before that we can view a closed n-walk in G as beginning and ending at any of its vertices, and furthermore that such a cycling of the vertices does not change the weight of the closed n-walk. We also note that if the matrices M_i are symmetric, then we may reverse the direction of each closed walk in the G, and that reversing direction does not affect the weight of the n-walk. These observations imply

Theorem 10 *Let $M_1, M_2, \ldots, M_d$ be square matrices all of the same size, and let $c_1, c_2, \ldots, c_n$ be a sequence in which each c_k is an integer with $1 \leq c_k \leq d$. Then*

$$\mathrm{Tr}(M_{c_1} M_{c_2} \ldots M_{c_n}) \quad = \quad \mathrm{Tr}(M_{c_{\sigma(1)}} M_{c_{\sigma(2)}} \cdots M_{c_{\sigma(n)}}) \qquad (1)$$

for any cyclic permutation σ of n letters. Furthermore, if each M_i is symmetric, then equation (1) holds for any dihedral permutation σ.

Again, we remark that the first part of Theorem 10, at least, follows immediately from the well-known Theorem 1. However, our graph-theoretic approach helps provide some intution about the traces of products and the significance of cyclic and dihedral permutations in this setting.

References

[1] Howard Anton. *Elementary Linear Algebra.* John Wiley & Sons, New York NY, seventh edition, 1994.

[2] Sheldon Axler. *Linear Algebra Done Right.* Springer, New York NY, 1996.

[3] Norman Biggs. *Algebraic Graph Theory.* Cambridge University Press, Cambridge, second edition, 1993.

[4] Charles W. Curtis. *Linear Algebra: An Introductory Approach.* Springer-Verlag, New York NY, 1984.

[5] Susanna S. Epp. *Discrete Mathematics With Applications.* PWS, Boston MA, second edition, 1995.

[6] Kenneth Hoffman and Ray Kunze. *Linear Algebra.* Prentice-Hall, Englewood Cliffs NJ, second edition, 1971.

[7] Serge Lang. *Linear Algebra.* Addison-Wesley, Reading MA, 1966.