Envelopes and String Art

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A familiar-looking problem

For each $\alpha \in [0,1]$ let ℓ_α be the line segment in $\mathbb{R}^2$ connecting the point $(\alpha, 0)$ with the point $(0, 1-\alpha)$. FIGURE 1 shows the segments ℓ_α for α equal to integer multiples of $1/20$.

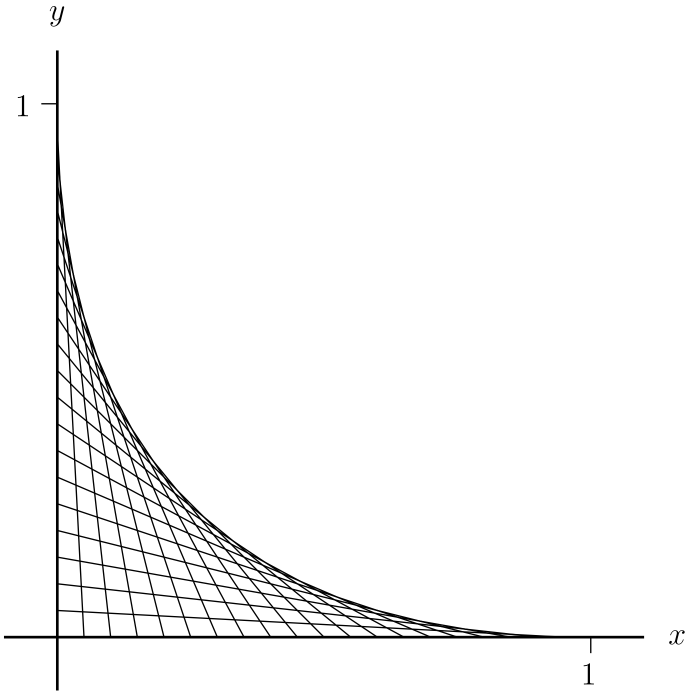

Figure 1: Line segments connecting $(\alpha, 0)$ with $(0, 1-\alpha)$

The upper right edge of this collection suggests a curve C from $(1,0)$ to $(0,1)$. This curve is an *envelope* of the collection $\{\ell_\alpha : 0 \le \alpha \le 1\}$, and has the property that each of its tangent lines contains one of segments ℓ_α. We'd like to find an equation for C.

Although C is characterized in terms of tangent lines, and thus is a solution to a differential equation, it turns out that we can determine C in a quite elementary way. The key is to recognize that when α and β are close together, the intersection point of ℓ_α and ℓ_β is close to C, and as β approaches α, the intersection point of ℓ_α and ℓ_β approaches a point of C, as in FIGURE 2.

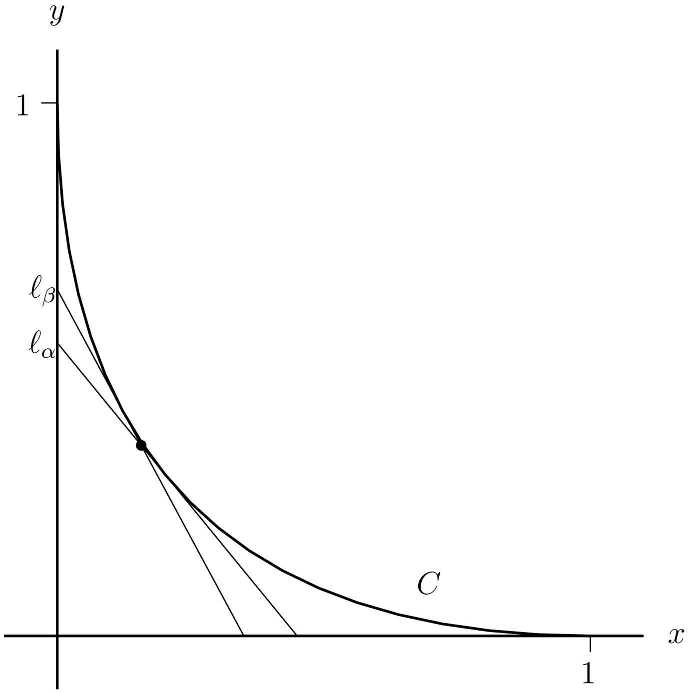

Figure 2: When α is close to β, the segments ℓ_α and ℓ_β intersect near C

To get the calculations going, we parametrize each segment ℓ_α as

$$\begin{aligned}\ell_\alpha(t) &= (1-t)(\alpha,0)+t(0,1-\alpha)\\ &= ((1-t)\alpha, t(1-\alpha)), \quad 0\le t\le 1.\end{aligned}$$

If $\beta \neq \alpha$, then ℓ_α and ℓ_β intersect at the point

$$\ell_\alpha(1-\beta)=\ell_\beta(1-\alpha)=(\alpha\beta,(1-\alpha)(1-\beta)).$$

To get a point on C, we want to take the limit of this last expression as $\beta \to \alpha$. Even a common-sense notion of limits will suffice here: we simply substitute α for β to find that the point $(\alpha^2, (1-\alpha)^2)$ lies on C. Just like that, we have a parametrization for C:

$$x = \alpha^2 \quad \text{and} \qquad y = (1-\alpha)^2. \tag{1}$$

With $0 \le \alpha \le 1$, both α and $1-\alpha$ are nonnegative, so we can write (1) as

$$\sqrt{x} + \sqrt{y} \;\; = \;\; 1. \tag{2}$$

A popular (in one sense) problem in calculus textbooks asks the student to show that the sum of the x- and y-intercepts of the tangent lines to the curve $\sqrt{x} + \sqrt{y} = \sqrt{c}$ is always equal to c [4, p. 234, problem 38]. Here we have solved what appears to be a more difficult problem—finding a curve whose tangent lines have intercepts with a constant sum—and we have done so with only the merest hint of the calculus.

A curve with equation $|ax|^p + |by|^p = c^p$, where $0 < p < 2$, is called a *hypoellipse* [6]. Since the equation of our envelope C has this form with $a = b = c = 1$, we can describe C as one quarter of the unit hypocircle with exponent $1/2$.

Eliminating the radicals from (2), we find that the points of C satisfy

$$x^2 + y^2 - 2xy - 2x - 2y + 1 \;\; = \;\; 0.$$

This is clearly a conic section, and since its discriminant is 0, it is a parabola. If we lift the restriction $0 \le \alpha \le 1$ and consider the ℓ_α as lines rather than just line segments, we find that the envelope of this family of lines is a parabola tangent to the coordinate axes and containing the first-quadrant part of the unit hypocircle. FIGURE 3 shows how all the pieces fit together.

String art

FIGURE 1 calls to mind the craft of string art, in which the artist creates a decorative pattern by driving nails into a board at intervals along a few lines

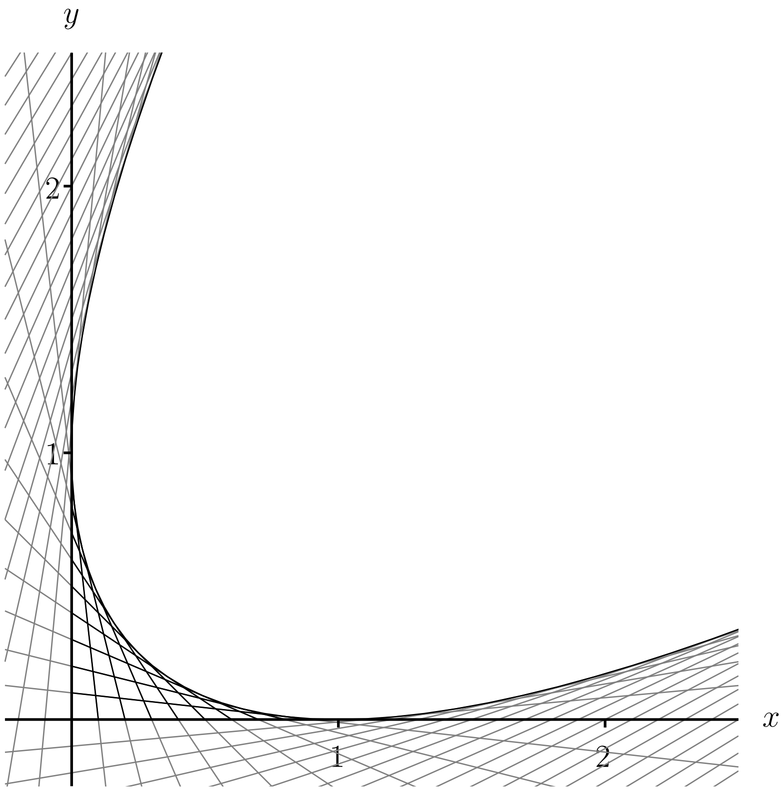

Figure 3: The parabola emerges when we draw the ℓ_α as lines rather than line segments and allow values of α outside $[0, 1]$

or curves and then connecting selected pairs of nails with stretched strings. The result of such an exercise is shown in FIGURE 4.

The very simple string art recipe with nails evenly spaced along the x- and y-axes and pairs chosen so that the sum of their x- and y-coordinates is constant gives the pattern in FIGURE 1 with the upper right edge approximating a branch of a hypocircle.

Evenly-spaced nails along perpendicular lines go only so far as an outlet for creative energy, and the ambitious string artist will surely want to experiment with different spacings, nonperpendicular lines, and nonlinear curves.

Likewise, the mathematician will ask what sort of envelopes arise when we vary the recipe for the parametrized family $\{\ell_\alpha\}$. In the following sections, we

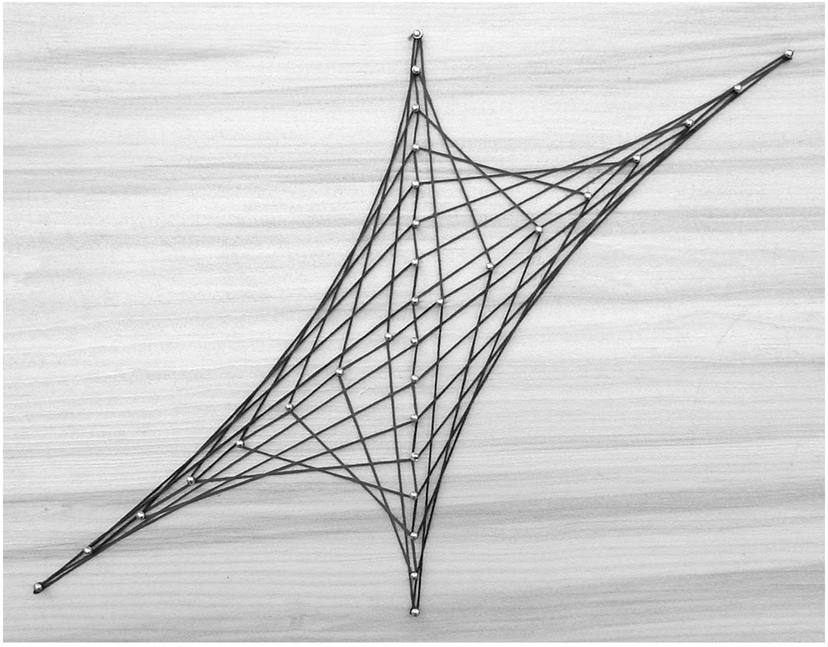

Figure 4: An unpretentious (to say the least) example of string art

experiment with some of these string-art variations, and see what envelope curves we get.

Spacing functions

First, let's keep the nails on the x- and y-axes, but change the way we space them. Each line ℓ_α will be determined by two points, $(X(\alpha), 0)$ and $(0, Y(\alpha))$, where X and Y are "spacing functions", which we will assume to be differentiable.

Again, we begin by finding the intersection point of two lines in this family. A straightforward calculation shows that ℓ_α and ℓ_β intersect at the point

$$\left(\frac{X(\alpha)X(\beta)(Y(\beta)-Y(\alpha))}{X(\alpha)Y(\beta)-Y(\alpha)X(\beta)}, \frac{Y(\alpha)Y(\beta)(X(\alpha)-X(\beta))}{X(\alpha)Y(\beta)-Y(\alpha)X(\beta)}\right). \tag{3}$$

To find a point on the envelope of the family ℓ_α, we take the limit of (3) as $\beta \to \alpha$. We use the standard trick of adding and subtracting $X(\alpha)Y(\alpha)$ in the denominator and then introducing some $(\beta - \alpha)$s to form difference quotients. For the x-coordinate of the envelope point, we get

$$\begin{aligned} x &= \lim_{\beta\to\alpha} \frac{X(\alpha)X(\beta)(Y(\beta)-Y(\alpha))}{X(\alpha)(Y(\beta)-Y(\alpha))-Y(\alpha)(X(\beta)-X(\alpha))} \\ &= \lim_{\beta\to\alpha} \frac{X(\alpha)X(\beta)\frac{Y(\beta)-Y(\alpha)}{\beta-\alpha}}{X(\alpha)\frac{Y(\beta)-Y(\alpha)}{\beta-\alpha} - Y(\alpha)\frac{X(\beta)-X(\alpha)}{\beta-\alpha}}. \end{aligned} \tag{4}$$

Now X and Y were assumed differentiable, so the mean value theorem says that the limit in (4) is equal to

$$\frac{(X(\alpha))^2 Y'(\alpha)}{X(\alpha)Y'(\alpha) - Y(\alpha)X'(\alpha)}. \tag{5}$$

A similar calculation gives the y-coordinate of a point on the envelope as

$$\frac{-(Y(\alpha))^2 X'(\alpha)}{X(\alpha)Y'(\alpha) - Y(\alpha)X'(\alpha)}. \tag{6}$$

Expressions (5) and (6) parametrize the envelope of the family ℓ_α determined by any pair of spacing functions X and Y.

Linear spacing functions: the details

In our first example, we used the particularly simple spacing functions

$$X(\alpha) = \alpha \quad \text{and} \quad Y(\alpha) = 1 - \alpha.$$

Let's look at what we get for an envelope when X and Y are general linear functions

$$X(\alpha) = r\alpha + h \quad \text{and} \quad Y(\alpha) = s\alpha + k,$$

with r and s nonzero. In this case, (5) and (6) yield the parametrization

$$x = \frac{(r\alpha + h)^2 s}{sh - rk} \quad \text{and} \quad y = -\frac{(s\alpha + k)^2 r}{sh - rk}, \tag{7}$$

provided that $sh \neq rk$. (If $sh = rk$, then the lines ℓ_α are parallel, and there is no envelope.)

What sort of curve is this? Guided by our earlier calculations, we begin experimenting with $\sqrt{|x|}$ and $\sqrt{|y|}$, and eventually find that the x and y in (7) satisfy

$$\sqrt{|sx|} + \sqrt{|ry|} \quad = \quad \frac{|(r\alpha + h)s|}{\sqrt{|sh - rk|}} + \frac{|(s\alpha + k)r|}{\sqrt{|sh - rk|}}. \tag{8}$$

If $(r\alpha+h)s$ and $(s\alpha+k)r$ have different signs, then the right side of (8) reduces to $\sqrt{|sh - rk|}$. Thus the points of our envelope curve for which $(r\alpha + h)s$ and $(s\alpha + k)r$ have different signs lie on the hypoellipse $\sqrt{|sx|} + \sqrt{|ry|} = \sqrt{|sh - rk|}$.

To see that we have a parabola as well, we need only verify (a tedious but straightforward task) that the parametrization in (7) satisfies the equation

$$s^2x^2 + r^2y^2 + 2rsxy - 2(sh - rk)(sx - ry) + (sh - rk)^2 = 0 \tag{9}$$

for every value of α.

As in our introductory example, this parabola is tangent to both the coordinate axes. The points of tangency are $(0, (rk - sh)/r)$ and $((sh - rk)/s, 0)$.

As a simple example, take $X(\alpha) = 2\alpha+1$ and $Y(\alpha) = \alpha+2$. The envelope (Figure 5) is parametrized by

$$x = -\frac{(2\alpha + 1)^2}{3} \quad \text{and} \quad y = \frac{2(\alpha + 2)^2}{3}.$$

It is tangent to the coordinate axes at the points $(0, 3/2)$ (where $\alpha = -1/2$) and $(-3, 0)$ (where $\alpha = -2$). Between the two points of tangency, the points on the envelope satisfy $\sqrt{|x|} + \sqrt{|2y|} = \sqrt{3}$.

The reader with a taste for the classical theory of conic sections might want to verify that the focus and directrix of the parabola given by (9) are

$$\left(\frac{s(sh - rk)}{r^2 + s^2}, -\frac{r(sh - rk)}{r^2 + s^2} \right) \quad \text{and} \quad rx - sy = 0.$$

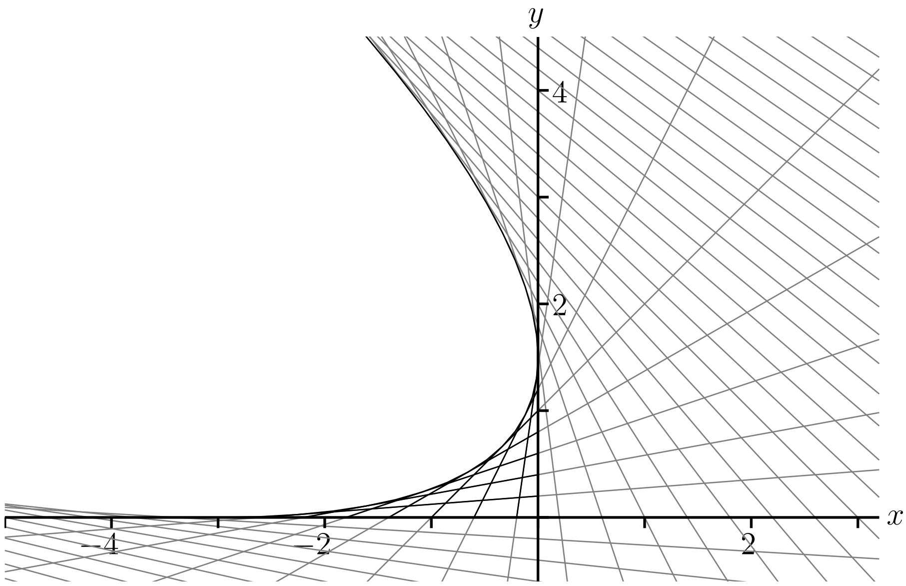

Figure 5: With $X(\alpha) = 2\alpha + 1$ and $Y(\alpha) = 2 + \alpha$, the envelope is a parabola in the second quadrant. The quarter hypoellipse is tangent to the ℓ_α with $-2 \leq \alpha \leq -1/2$.

Off the coordinate axes

Suppose now that the string artist chooses (nonparallel) nailing lines n_1 and n_2 other than the coordinate axes. The mathematician may then straighten out this skewed situation by cooking up an affine transformation $\mathcal{A}$ that takes the x-axis to line n_1 and the y-axis to line n_2.

Assuming that the string artist is still spacing the nails evenly, the mathematician can find spacing functions $X(\alpha) = r\alpha + h$ and $Y(\alpha) = s\alpha + k$ such that $\mathcal{A}(X, 0)$ and $\mathcal{A}(0, Y)$ agree with the artist's nailing pattern. The envelope curve on the string art will then be the image under $\mathcal{A}$ of the parabolic curve that we found earlier. Since a nonsingular affine transformation takes parabolas to parabolas, an envelope curve that arises from evenly-spaced nails along any two lines n_1 and n_2 will also lie on a parabola, tangent to lines n_1 and n_2.

Thus the envelope curves in FIGURE 4, which might at first glance suggest a pair of hyperbolas, are in fact parts of four parabolas, pairwise tangent at the ends of the nailing lines.

For a different sort of illustration, we make a short digression into game theory. Consider a two-player, non-zero-sum game in which each player has two pure strategies available. We can represent such a game using a table like this one:

		Player II	
		A	B
Player I	A	$(2, 0)$	$(3, 6)$
	B	$(4, 2)$	$(0, 0)$

The ordered pair $(2, 0)$ in the upper left corner means that if Player I chooses strategy IA and Player II chooses strategy IIA, then the payoff to Player I is 2 and the payoff to Player II is 0.

Now suppose that the game is played repeatedly. For each play, Player I uses some random device to select a strategy. Suppose she chooses strategy IA with probability α and IB with probability $1 - \alpha$. Similarly, Player II chooses strategy IIA with probability β and IIB with probability $1 - \beta$. For this repeated play, we can calculate an *expected payoff*, since we know the probability with which each of the four payoff pairs will occur. The expected payoff is

$$\alpha\beta(2, 0) + \alpha(1 - \beta)(3, 6) + (1 - \alpha)\beta(4, 2) + (1 - \alpha)(1 - \beta)(0, 0).$$

We factor this to get

$$(1 - \beta)(\alpha(3, 6) + (1 - \alpha)(0, 0)) + \beta(\alpha(2, 0) + (1 - \alpha)(4, 2)). \tag{10}$$

Since $0 \leq \beta \leq 1$, expression (10) shows that the expected payoff, considered as a point in $\mathbb{R}^2$, lies on the line segment connecting $\alpha(2, 0) + (1 - \alpha)(4, 2)$ with $\alpha(3, 6) + (1 - \alpha)(0, 0)$. We denote this line segment ℓ_α and observe that the set of possible expected payoffs is equal to the union of all the segments ℓ_α, $0 \leq \alpha \leq 1$.

This is the shaded region in FIGURE 6. The curved part of its boundary is an envelope of exactly the kind we have been discussing, so it must lie on a parabola.

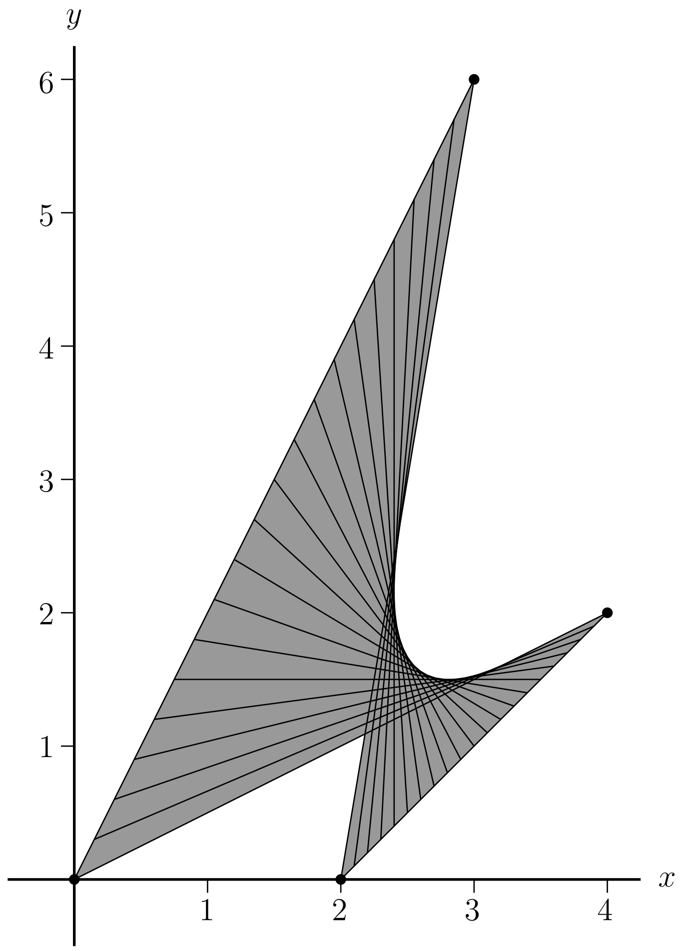

Figure 6: Possible expected payoffs in a non-zero-sum game

To find an explicit parametrization for this parabola, we straighten things out with the affine map $\mathcal{A} : (x, y) \mapsto (x + y - 2, x + 2y - 4)$. This map takes the portion of the x-axis between $x = 4$ and $x = 6$ to the line segment connecting $(2, 0)$ with $(4, 2)$ and the portion of the y-axis between $y = 2$ and

$y = 5$ to the line segment connecting the origin with $(3, 6)$. We take

$$X(\alpha) = 4 + 2\alpha \quad \text{and} \quad Y(\alpha) = 5 - 3\alpha$$

and apply (7) to get the parabola parametrized by

$$x = \frac{3}{22}(4 + 2\alpha)^2 \quad \text{and} \quad y = \frac{1}{11}(5 - 3\alpha)^2.$$

The curved part of the boundary of the shaded region in FIGURE 6 is the image of this parabola under $\mathcal{A}$. Its parametrization is

$$x = \frac{3}{11}(5\alpha^2 - 2\alpha + 9) \quad \text{and} \quad y = \frac{6}{11}(4\alpha^2 - 6\alpha + 5),$$

with $0 \leq \alpha \leq 1$.

The shaded region shows the set of expected payoffs that can arise if each player uses a random device to choose a strategy each time the game is played. If the game is played a large number of times and the average payoff converges to a point outside the shaded region, then we have evidence that the players' random devices are *not independent*. In certain circumstances, this might indicate collusion, espionage, or just poor random number generators.

Envelopes from right triangles

We now move our nailing lines back to the coordinate axes and consider some specific non-linear spacing functions.

Constant area

Let $X(\alpha) = ke^{\alpha}$ and $Y(\alpha) = ke^{-\alpha}$ for some non-zero constant k. Then the lines ℓ_α have the property that the product of the x- and y-intercepts of each line is equal to k^2. Put another way, the ℓ_α are the hypotenuses of a family of right triangles with constant area.

Applying formulas (5) and (6), we find that the envelope of this family of lines is parametrized by

$$x = \frac{ke^{\alpha}}{2} \quad \text{and} \quad y = \frac{ke^{-\alpha}}{2}.$$

FIGURE 7 shows the envelope of this family of lines. This time the envelope *is* a branch of a hyperbola; its equation is $xy = k^2/4$.

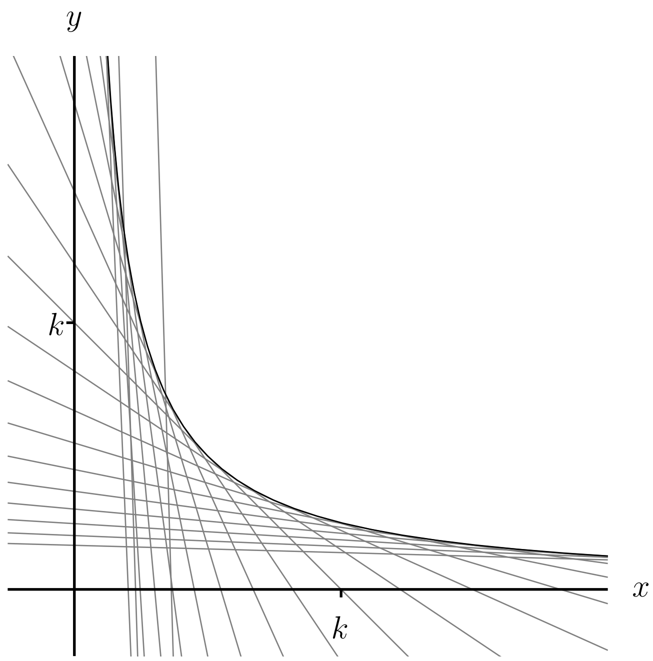

Figure 7: With $X = ke^{\alpha}$ and $Y = ke^{-\alpha}$, the envelope is the hyperbola $xy = k^2/4$

Constant length: the calculus of ladders

Consider a ladder of fixed length L sliding down a wall. The ladder describes a family of lines, shown in FIGURE 8, for which the distance between the x-intercept $(X(\alpha), 0)$ and the y-intercept $(0, Y(\alpha))$ is constantly equal to L.

We want to choose X and Y so that $X(\alpha)^2 + Y(\alpha)^2 = L^2$. An obvious choice is

$$X(\alpha) = L\cos\alpha \quad \text{and} \quad Y(\alpha) = L\sin\alpha.$$

Formulas (5) and (6) give the envelope of this set of lines as

$$x = L\cos^3\alpha \quad \text{and} \quad y = L\sin^3\alpha. \tag{11}$$

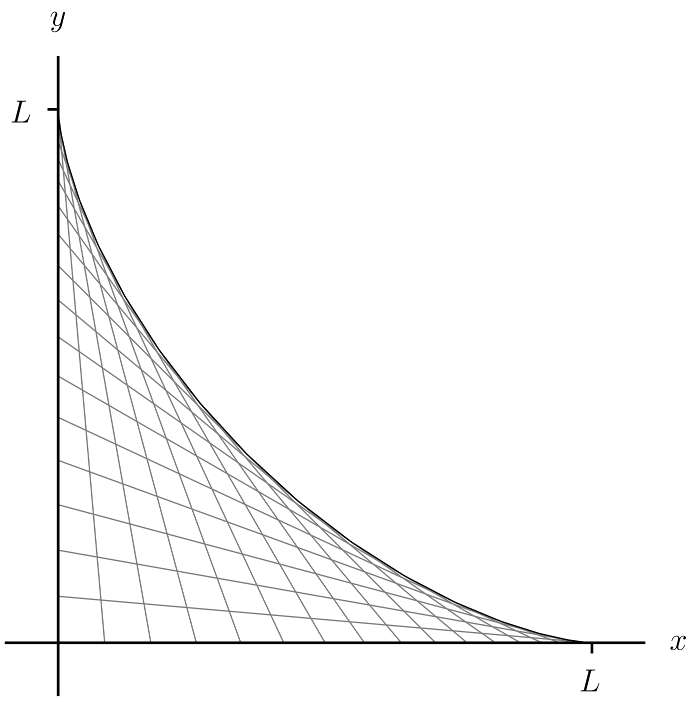

Figure 8: The sliding ladder's envelope lies along the astroid $x^{2/3}+y^{2/3} = L^{2/3}$

A Cartesian equation for this curve is $x^{2/3} + y^{2/3} = L^{2/3}$; it is called an astroid.

Applying a handful of trig identities, one can rewrite (11) as

$$x = \frac{3L}{4}\cos\alpha + \frac{L}{4}\cos(4\alpha) \quad \text{and} \quad y = \frac{3L}{4}\sin\alpha - \frac{L}{4}\sin(4\alpha),$$

showing that the sliding-ladder envelope also happens to lie on the hypocycloid traced by a point on a circle of radius $L/4$ rolling along the inside of a circle of radius L.

Want to carry your ladder around a corner from one hallway into another? If the widths of the hallways are x and y, then the astroid equation above shows that x, y, and L have to satisfy $x^{2/3} + y^{2/3} \geq L^{2/3}$ in order for the ladder to make it around the corner horizontally.

Constant perimeter

Let r be a positive constant. Let

$$X(\alpha) = r - \alpha \quad \text{and} \quad Y(\alpha) = \frac{2r\alpha}{r+\alpha}$$

for $0 \leq \alpha \leq r$. It is not hard to check that

$$X(\alpha) + Y(\alpha) + \sqrt{(X(\alpha))^2 + (Y(\alpha))^2} = 2r.$$

That is, the triangles with vertices at the origin, $(X(\alpha), 0)$, and $(0, Y(\alpha))$ all have perimeter $2r$.

We use (5) and (6) to find a parametrization for the envelope of the hypotenuses of these triangles. We get

$$x = \frac{r(r-\alpha)^2}{r^2+\alpha^2} \quad \text{and} \quad y = \frac{2r\alpha^2}{r^2+\alpha^2}.$$

A little algebra shows that the x and y in this parametrization satisfy $(x - r)^2 + (y-r)^2 = r^2$ for all α. This envelope is part of a circle. Thus if we have a loop of string with length $2r$ and we stretch it into a triangle with a right angle at the origin and the legs along the positive x- and y-axes, the hypotenuse of the triangle will be tangent to the circle with center (r, r) and radius r. FIGURE 9 shows the circle determined by these taut-string triangles.

Free-form nailing

We conclude by considering a generalization in which the nailing lines become arbitrary curves in the plane.

Let $C_1(\alpha) = (X_1(\alpha), Y_1(\alpha))$ and $C_2(\alpha) = (X_2(\alpha), Y_2(\alpha))$ be two differentiable curves. For each value of α, we can parametrize the line ℓ_α determined by $C_1(\alpha)$ and $C_2(\alpha)$ as

$$\ell_\alpha(t) \quad = \quad (1-t)(X_1(\alpha), Y_1(\alpha)) + t(X_2(\alpha), Y_2(\alpha)). \tag{12}$$

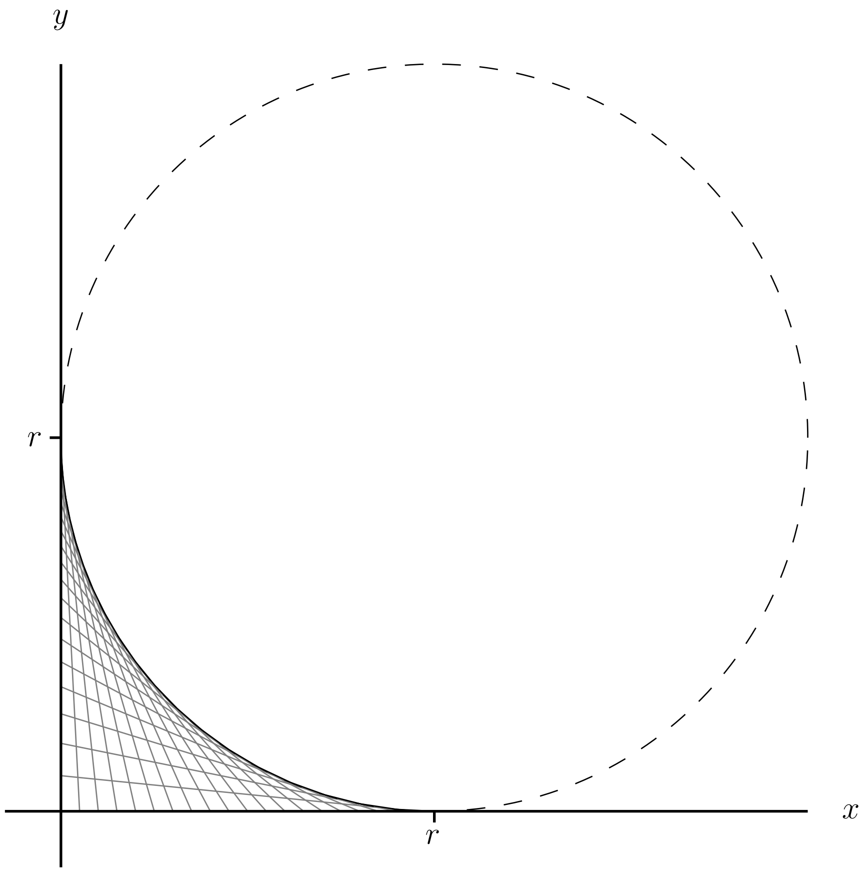

Figure 9: Triangles with a right angle at the origin and perimeter $2r$ are all tangent to the circle $(x-r)^2+(y-r)^2=r^2$

Then for $\alpha \neq \beta$ the lines ℓ_α and ℓ_β (assuming they aren't parallel) intersect when t has the value

$$\frac{(X_2(\alpha)-X_1(\alpha))(Y_1(\beta)-Y_1(\alpha))-(Y_2(\alpha)-Y_1(\alpha))(X_1(\beta)-X_1(\alpha))}{(X_2(\beta)-X_1(\beta))(Y_2(\alpha)-Y_1(\alpha))-(Y_2(\beta)-Y_1(\beta))(X_2(\alpha)-X_1(\alpha))}. \quad (13)$$

To find the point where ℓ_α is tangent to the envelope of the family $\{\ell_\alpha\}$, we'll need to find the limit of (13) as $\beta \to \alpha$. The factors $(Y_1(\beta)-Y_1(\alpha))$ and $(X_1(\beta)-X_1(\alpha))$ in the numerator of (13) can be rewritten as $(\beta-\alpha)$ times the obvious difference quotients. With a little algebraic sleight of hand, the denominator or (13) can be put in the form

$$
\begin{array}{l}
[(X_2(\beta) - X_2(\alpha)) - (X_1(\beta) - X_1(\alpha))](Y_2(\alpha) - Y_1(\alpha)) \\
-[(Y_2(\beta) - Y_2(\alpha)) - (Y_1(\beta) - Y_1(\alpha))](X_2(\alpha) - X_1(\alpha)).
\end{array}
$$

Again, the first factor in each term is $(\beta - \alpha)$ times an appropriate difference quotient. In the limit, the difference quotients become derivatives and we get

$$
\lim_{\beta \to \alpha} t = \frac{(X_2 - X_1)Y_1' - (Y_2 - Y_1)X_1'}{(X_2' - X_1')(Y_2 - Y_1) - (Y_2' - Y_1')(X_2 - X_1)}, \tag{14}
$$

where all the functions on the right are evaluated at α. We evaluate (12) at this value of t to find a point (x, y) on the envelope. We get (evaluating everything at α)

$$
x = \frac{(X_1X_2' - X_1'X_2)(Y_2 - Y_1) - (X_1Y_2' - Y_1'X_2)(X_2 - X_1)}{(X_2' - X_1')(Y_2 - Y_1) - (Y_2' - Y_1')(X_2 - X_1)} \tag{15}
$$

$$
y = \frac{(Y_1X_2' - X_1'Y_2)(Y_2 - Y_1) - (Y_1Y_2' - Y_1'Y_2)(X_2 - X_1)}{(X_2' - X_1')(Y_2 - Y_1) - (Y_2' - Y_1')(X_2 - X_1)} \tag{16}
$$

as the generalization of (5) and (6).

Here's another route to (15) and (16) that takes a brief detour through 3-space. This may look more familiar to a differential geometer.

We put our first curve $(X_1(\alpha), Y_1(\alpha))$ in the plane $z = 0$ and our second curve $(X_2(\alpha), Y_2(\alpha))$ in the plane $z = 1$. Then we stretch the strings from one curve to the other through the interjacent slice of $\mathbb{R}^3$ to form a surface (a ruled surface, in fact) with the parametrization

$$
x(\alpha, t) = X_1(\alpha) + t(X_2(\alpha) - X_1(\alpha)) \tag{17}
$$

$$
y(\alpha, t) = Y_1(\alpha) + t(Y_2(\alpha) - Y_1(\alpha)) \tag{18}
$$

$$
z(\alpha, t) = t,
$$

with $0 \leq t \leq 1$. If we look straight down on this surface, then the strings we have stretched between the plane $z = 0$ and $z = 1$ look just like our line segments ℓ_α, and their envelope is the "visual edge" of the surface, viewed from directly overhead.

A point on the visual edge of the surface is distinguished by the fact that the normal vector there is horizontal. The normal vector to our surface has a z-component given by $(\partial x/\partial\alpha)(\partial y/\partial t) - (\partial y/\partial\alpha)(\partial x/\partial t)$, so a point (α, t) maps to a point on the visual edge when

$$\frac{\partial x}{\partial\alpha}(\alpha,t)\frac{\partial y}{\partial t}(\alpha,t) \;=\; \frac{\partial y}{\partial\alpha}(\alpha,t)\frac{\partial x}{\partial t}(\alpha,t). \tag{19}$$

Using the parametrization (17) and (18), condition (19) becomes

$$\begin{aligned} &X_1'(Y_2 - Y_1) + t(X_2' - X_1')(Y_2 - Y_1) \\ &\quad = \; Y_1'(X_2 - X_1) + t(Y_2' - Y_1')(X_2 - X_1), \end{aligned} \tag{20}$$

with everything evaluated at α.

Solving (20) for t gives us exactly the value we found in (14), and when we substitute that value into (17) and (18), we recover the parametrization (15) and (16).

Let's try out formulas (15) and (16) on some simple parametrized curves: two circles centered at the origin.

Let $X_1(\alpha) = 2\sin\alpha$, $Y_1(\alpha) = 2\cos\alpha$, $X_2(\alpha) = -\sin\alpha$, and $Y_2(\alpha) = \cos\alpha$. Two points determine each of our lines ℓ_α. The first moves clockwise around a circle of radius 2, starting at the 12 o'clock position; the second moves counterclockwise around a circle of radius 1, also starting at 12 o'clock. Unlike the hands of an actual clock, our two points move at the same angular rate.

Applying (15) and (16) to these two circles, we get an envelope curve parametrized by

$$x = -4\sin^3\alpha \quad \text{and} \quad y = \frac{4}{3}\cos^3\alpha. \tag{21}$$

The obvious quantity to compute here is $x^{2/3} + (3y)^{2/3}$, and we find that the points along (21) satisfy

$$x^{\frac{2}{3}} + (3y)^{\frac{2}{3}} \;=\; 4^{\frac{2}{3}}.$$

Our envelope is another hypoellipse. In fact, since the exponent is 2/3, we might also describe this envelope as a sort of squashed astroid.

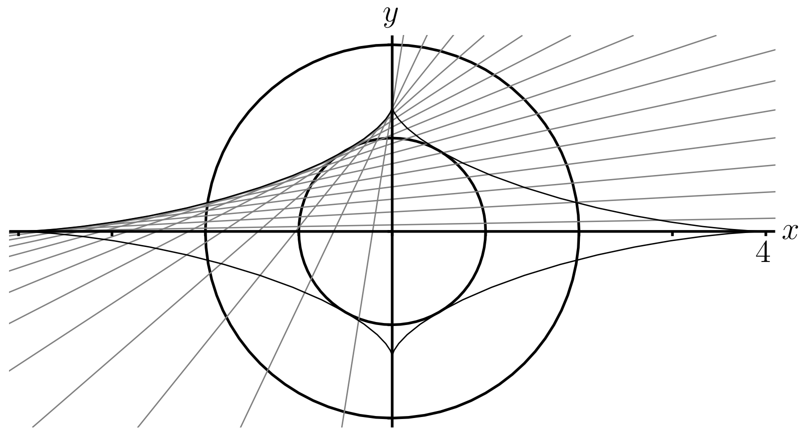

Figure 10: The lines determined by one point moving clockwise at radius 2 and another moving counterclockwise at radius 1 are all tangent to a squashed astroid

Figure 10 shows the two paths, the envelope, and the lines ℓ_α that generate just the first quarter of the envelope. Since the points of tangency along most of this envelope lie outside the segments "between the nails," a traditional string art piece using this recipe would show only a small part of the envelope. To get the whole picture, as Figure 10 suggests, we might want to try a kind of augmented string art, in which we stretch the lines all the way to the frame, and anchor them there.

Other envelopes that arise from points moving around circles will be found in [2] and [3].

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