Notes on an example of McLaughlin

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1 Introduction

In his 1986 doctoral thesis [4], John McLaughlin computes the spectrum of the adjacency operator on the Cayley graph of the free product $\mathbb{Z}/2\mathbb{Z}\star\mathbb{Z}/3\mathbb{Z}$. This example involves a nice mixture of combinatorics, spectral theory of finite-dimensional operators, and spectral theory of infinite-dimensional operators. In [5], we examined the finite-dimensional and combinatorial phases of McLaughlin's computations. Here, we turn to the next phase, and try to understand in fairly elementary terms how to use the spectral theorem to turn combinatorial information about an infinite-dimensional operator into a description of the operator's spectrum and spectral measures.

2 Graphs and operators

Let Γ be an undirected, k-regular graph. A *function* on Γ is a map from the vertex set of Γ to the complex numbers. The notation $x \in \Gamma$ means x is a vertex of Γ, and $x \sim y$ means x and y are joined by an edge. With the inner product

$$\langle f, g\rangle = \sum_{x\in\Gamma} f(x)\overline{g(x)}$$

the space $L^2(\Gamma)$ of square-summable functions on Γ becomes a Hilbert space. If Γ has a finite number N of vertices, then $L^2(\Gamma) = \mathbb{C}^N$. If Γ is infinite, then $L^2(\Gamma) = \ell^2$.

For each $x \in \Gamma$, let $\delta_x : \Gamma \to \mathbb{C}$ be the function

$$\delta_x(y) = \begin{cases} 1 & \text{if } y = x \\ 0 & \text{otherwise.} \end{cases}$$

Then $\{\delta_x : x \in \Gamma\}$ is an orthonormal basis for $L^2(\Gamma)$, which we shall call the *standard basis*.

The adjacency operator $A : L^2(\Gamma) \to L^2(\Gamma)$ is given by

$$Af(x) = \sum_{y \sim x} f(y).$$

The fact that A is self-adjoint follows from the fact that Γ is undirected:

$$\begin{aligned} \langle A\delta_x, \delta_y \rangle &= A\delta_x(y) \\ &= \begin{cases} 1 & \text{if } y \sim x \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

while

$$\begin{aligned} \langle \delta_x, A\delta_y \rangle &= \overline{A\delta_y(x)} \\ &= \begin{cases} 1 & \text{if } x \sim y \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

The fact that A is bounded follows from the regularity of Γ.

Claim 1 *If A is the adjacency operator on $L^2(\Gamma)$, where Γ is a k-regular graph, then $||A|| \leq k$.*

Proof Let $f \in L^2(\Gamma)$ with $||f|| = 1$. Then

$$\langle Af, f \rangle = \sum_{x \in \Gamma} \sum_{y \sim x} f(x)\overline{f(y)}. \quad (1)$$

The sum in (1) contains a term $f(x)\overline{f(y)}$ and a term $f(y)\overline{f(x)}$ for every pair of adjacent vertices x and y in Γ. If we think of each (undirected) edge in Γ as a pair of oppositely-oriented directed edges, then the right side of (1) may be viewed as a sum over directed edges. If $\vec{a}$ is a directed edge in Γ, let $\cdot\vec{a}$ and $\vec{a}\cdot$ denote the initial and terminal vertices of $\vec{a}$. Then we have

$$\langle Af, f\rangle \;=\; \sum_{\vec{a}} f(\cdot\vec{a})\overline{f(\vec{a}\cdot)}. \tag{2}$$

Now by the Schwarz inequality,

$$\left|\sum_{\vec{a}} f(\cdot\vec{a})\overline{f(\vec{a}\cdot)}\right| \;\le\; \left(\sum_{\vec{a}} |f(\cdot\vec{a})|^2 \sum_{\vec{a}} |f(\vec{a}\cdot)|^2\right)^{\frac{1}{2}}.$$

Since each vertex in Γ is the initial vertex for k edges and the terminal vertex for k edges and $||f|| = 1$, we have

$$\sum_{\vec{a}} |f(\cdot\vec{a})|^2 = \sum_{\vec{a}} |f(\vec{a}\cdot)|^2 = k\sum_{x\in\Gamma} |f(x)|^2 = k.$$

Thus $|\,\langle Af, f\rangle\,| \le k$, and the claim is proved. □

3 The return generating function

Let x and y be vertices in a graph Γ. A *walk of length* n from x to y is a sequence of $n+1$ vertices $x_0, x_1, \ldots, x_n$ such that $x_0 = x$, $x_n = y$, and $x_{i-1} \sim x_i$ for $i = 1, \ldots n$. The fundamental connection between the operator A and the geometry of Γ (see [1]) is found in

Theorem 2 *The number of walks of length n from a vertex x to a vertex y in a graph Γ with adjacency operator A is equal to $\langle \delta_x, A^n\delta_y\rangle$.*

Proof For $n = 0$ and $n = 1$, the result is clear. Now suppose $n > 1$. Any walk of length n from x to y must have as its next-to-last vertex a neighbor of y. Thus by induction we get

$$\left(\begin{array}{c}\text{number of length-}n\\ \text{walks from } x \text{ to } y\end{array}\right) \;=\; \sum_{z\sim y}\left(\begin{array}{c}\text{number of length-}(n-1)\\ \text{walks from } x \text{ to } z\end{array}\right)$$

$$
\begin{aligned}
&= \sum_{z \sim y} \left\langle \delta_x, A^{n-1} \delta_z \right\rangle \\
&= \left\langle \delta_x, A^{n-1} \sum_{z \sim y} \delta_z \right\rangle \\
&= \left\langle \delta_x, A^{n-1} A \delta_y \right\rangle \\
&= \langle \delta_x, A^n \delta_y \rangle
\end{aligned}
$$

and the proof is complete. □

Let $x \in \Gamma$. A *closed walk of length* n *at* x is a walk of length n which begins and ends at x. The *return generating function at* x is a generating function which counts the number of closed walks at x. Denoting the function $R_x(z)$ and using the theorem above, we have

$$
R_x(z) \quad = \quad \sum_{n=0}^{\infty} \langle \delta_x, A^n \delta_x \rangle \, z^n. \tag{3}
$$

We remark that for $|z| < ||A||^{-1}$, the series in (3) converges. Since the adjacency operator A on a k-regular graph satisfies $||A|| \leq k$, any return generating function $R_x(z)$ on such a graph has a positive radius of convergence.

A graph Γ is called *vertex-transitive* if the automorphism group of Γ acts transitively on its set of vertices. It is immediate that a vertex-transitive graph is regular. Furthermore, in a vertex-transitive graph, the number of closed walks of length n at one vertex is the same as the number of closed walks of length n at any other vertex. Thus if Γ is vertex-transitive, we can speak of the return generating function $R(z)$ on Γ, without reference to any particular vertex x.

The return generating function on a finite, vertex-transitive graph is easily expressed in terms of the eigenvalues of the adjacency operator A. In [5] (see also [1], p. 13), we showed

Theorem 3 *The return generating function on a vertex-transitive graph with*

N vertices is given by

$$R(z) \quad = \quad \frac{1}{N}\sum_{i=1}^{N}\frac{1}{1-\lambda_i z}$$

where $\lambda_1, \lambda_2, \ldots, \lambda_N$ *are the eigenvalues of the graph's adjacency operator.*

4 McLaughlin's example

Let Γ_1 and Γ_2 be Cayley graphs for the groups G_1 and G_2 on (finite) sets of generators S_1 and S_2, respectively. To keep matters simple, choose S_1 and S_2 to be symmetric, so that Γ_1 and Γ_2 are undirected, regular, vertex-transitive graphs. Then the Cayley graph of the free product $G_1 \star G_2$ on the set of generators $S_1 \cup S_2$ is called the *free product* of Γ_1 and Γ_2 (or the *amalgam* of Γ_1 and Γ_2), and is denoted $\Gamma_1 \star \Gamma_2$. It is also an undirected, vertex-transitive, regular graph.

In [5], we gave a combinatorial proof of a result of McLaughlin ([4]) showing how the return generating function on $\Gamma_1 \star \Gamma_2$ can be computed (in theory, at least) from the return generating functions on Γ_1 and Γ_2. The result is

Theorem 4 *Let* Γ_1 *and* Γ_2 *be undirected Cayley graphs with return generating functions* $R_1(z)$ *and* $R_2(z)$ *respectively. The return generating function* $R(z)$ *on* $\Gamma_1 \star \Gamma_2$ *satisfies the three equations*

$$\begin{aligned} R(z) &= \frac{1}{1-S_1(z)-S_2(z)} \\ R(z) &= \frac{1}{1-S_1(z)}R_2\left(\frac{z}{1-S_1(z)}\right) \\ R(z) &= \frac{1}{1-S_2(z)}R_1\left(\frac{z}{1-S_2(z)}\right), \end{aligned}$$

where $S_1(z)$ *and* $S_2(z)$ *are functions which may be determined by solving the system.*

If Γ_1 and Γ_2 are Cayley graphs of finite groups, then Theorem 3 gives us explicit expressions for their return generating functions. If Γ_1 and Γ_2 are *really* small, then the system in Theorem 4 can be solved explicitly, and we can write down an expression for the return generating function on the infinite graph $\Gamma_1 \star \Gamma_2$. Since the return generating function for any graph is closely related to the graph's adjacency operator, we can then use this expression to glean information about the spectrum of the adjacency operator on $\Gamma_1 \star \Gamma_2$.

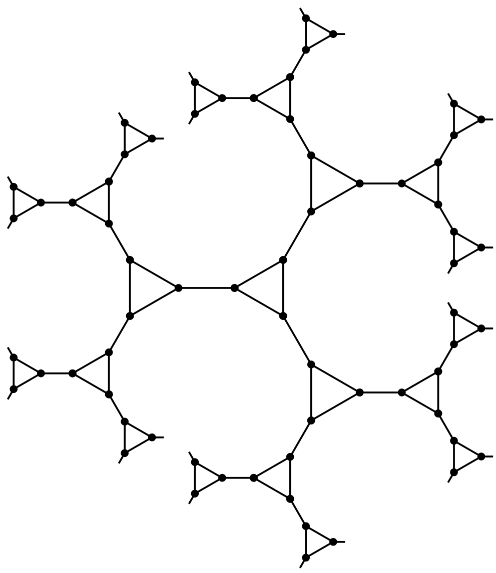

Figure 1: Part of the graph $\Gamma_1 \star \Gamma_2$

The example $G_1 = \mathbb{Z}/2\mathbb{Z}$, $G_2 = \mathbb{Z}/3\mathbb{Z}$ (with the natural – indeed the only possible – symmetric generating sets) represents one case where the return generating function equations can be solved. In [4], McLaughlin computed the spectrum of the adjacency operator on the Cayley graph of $G_1 \star G_2$ by first computing the return generating function. Gutkin generalized the computation somewhat in [2]. In [5], we used combinatorial methods to compute the return generating function for this graph. Our purpose in this paper is to understand the details of McLaughlin's spectral computation, and show, using elementary techniques, how to get from the return generating function to a complete spectral description of the adjacency operator.

Let Γ_1 and Γ_2 be the (undirected) Cayley graphs of $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/3\mathbb{Z}$ respectively, and let $R_1(z)$ and $R_2(z)$ be their return generating functions. Let

$\Gamma = \Gamma_1 \star \Gamma_2$ (see Figure 1), and let $R(z)$ denote its return generating function.

The adjacency operator on Γ_1 (a 2-by-2 matrix) has eigenvalues -1 and 1, and the adjacency operator on Γ_2 has eigenvalues -1, -1, and 2. Thus Theorem 3 gives us

$$R_1(z) = \frac{1}{1-z^2} \tag{4}$$

$$R_2(z) = \frac{z-1}{2z^2+z-1}. \tag{5}$$

The system in Theorem 4 becomes

$$\begin{aligned} R(z) &= \frac{1}{1-S_1(z)-S_2(z)} \\ &= \frac{1}{1-S_1(z)}\left[\frac{\frac{z}{1-S_1(z)}-1}{2\left(\frac{z}{1-S_1(z)}\right)^2+\frac{z}{1-S_1(z)}-1}\right] \\ &= \frac{1}{1-S_2(z)}\left[\frac{1}{1-\left(\frac{z}{1-S_2(z)}\right)^2}\right]. \end{aligned}$$

The solution to this system (see [5] for the details) is

$$R(z) = \frac{2(z-1)}{z^3+z^2-z+(z-2)\sqrt{z^4+6z^3-5z^2-2z+1}}.$$

We put spectral information (lists of eigenvalues) into the machinery of Theorem 4, and it handed us a combinatorial object, the generating function $R(z)$. We now turn to the problem of getting spectral information back out of this combinatorial generating function.

5 The spectral theorem for dummies

Let Γ denote the graph in Figure 1. The adjacency operator A on Γ is a bounded, self-adjoint, linear operator on an infinite-dimensional Hilbert space $L^2(\Gamma)$. The *spectrum* of such an operator is the set

$$\mathrm{Spec}(A) \quad = \quad \{\lambda : A - \lambda I \text{ is not invertible}\}.$$

It is well-known (see [3], for example) that $\mathrm{Spec}(A)$ is a compact subset of $\mathbb{R}$.

The relations between $\mathrm{Spec}(A)$ and the structure of A itself are given in the various forms of the spectral theorem. The statement of the spectral theorem which we will need here involves an algebra of operators generated by polynomials in A.

Briefly, if A is a bounded, self-adjoint, linear operator on a Hilbert space H and if p is a polynomial (with real coefficients), then $p(A)$ is also a bounded, self-adjoint, linear operator on H, so we can define a map T_A from the set of polynomials to the space $\mathcal{L}(H)$ of bounded, self-adjoint, linear operators on H by

$$T_A(p) \quad = \quad p(A).$$

The spectral theorem tells us how to extend the map T_A to the set $\mathcal{B}(\mathbb{R})$ of all real-valued Borel functions on $\mathbb{R}$. Here are the parts of the spectral theorem which we will need (see [6], p. 225).

Theorem 5 *Let A be a bounded self-adjoint linear operator on a Hilbert space H. Then there is a unique map $T_A : \mathcal{B}(\mathbb{R}) \to \mathcal{L}(H)$ satisfying*

1. *If $f(x) = x$ then $T_A(f) = A$.*
2. *T_A is an algebraic $*$-homomorphism. That is, $T_A(fg) = T_A(f)T_A(g)$; $T_A(\lambda f) = \lambda T_A(f)$ for constants λ; $T_A(1) = I$; $T_A(\overline{f}) = (T_A(f))^*$.*
3. *If $BA = AB$ then $BT_A(f) = T_A(f)B$ for any f.*
4. *If $f \geq 0$ then $T_A(f) \geq 0$.*

We get our hands on the map T_A through the *spectral measures*, as follows. Let φ be a vector in H. Then the map

$$f \quad \mapsto \quad \langle \varphi, T_A(f)\varphi \rangle \tag{6}$$

is a positive linear functional on the set of Borel functions on $\mathbb{R}$. In particular, we have a positive linear functional on the set of all continuous functions on $\mathbb{R}$ with compact support, so we can invoke the Riesz representation theorem ([7], p. 130) to produce a unique measure μ_φ satisfying

$$\langle \varphi, T_A(f)\varphi \rangle = \int f \, d\mu_\varphi$$

for each compactly-supported, continuous function f. The measure μ_φ is called the spectral measure associated with the vector φ.

It can be shown ([3]) that for each vector φ, the closure of the support of the spectral measure μ_φ is a subset of $\mathrm{Spec}(A)$. Furthermore, if we choose a large enough set of vectors φ, the support of the set of measures μ_φ will be equal to $\mathrm{Spec}(A)$.

More precisely, we have (again see [6])

Theorem 6 *If Φ is a set of vectors such that the span of the set*

$$\overline{\Phi} = \{A^n \varphi : \varphi \in \Phi, n = 0, 1, 2, \ldots\}$$

is dense in H, then the spectrum of A is equal to

$$\overline{\bigcup_{\varphi \in \Phi} \mathrm{supp}(\mu_\varphi)}.$$

In the case at hand, where $H = L^2(\Gamma)$ is the space of square-summable functions on a vertex-transitive graph, we take

$$\Phi = \{\delta_x : x \in \Gamma\}.$$

Since the span of Φ contains $L^2(\Gamma)$, the span of $\overline{\Phi}$ is certainly dense in $L^2(\Gamma)$, and so to find the spectrum of A we need only find the support of each measure μ_{δ_x}.

This task is simplified somewhat by making the observation that one μ_{δ_x} is pretty much like another.

Theorem 7 *Let Γ be a vertex-transitive graph with adjacency operator A. Let μ_{δ_x} and μ_{δ_y} be the spectral measures associated with the delta functions at x and y in Γ. Then $\mu_{\delta_x} = \mu_{\delta_y}$.*

Proof Since Γ is vertex-transitive, there is an automorphism P of Γ which takes x to y. Because P is a graph automorphism, it commutes with the adjacency operator, that is, $AP = PA$. (We are abusing notation here, and letting $P : L^2(\Gamma) \to L^2(\Gamma)$ by $P\varphi(x) = \varphi(Px)$.) Using T_A to denote the map in the spectral theorem, then, we get $PT_A(f) = T_A(f)P$ for any Borel function f. In particular, if $S \subset \mathbb{R}$ with characteristic function χ_S, then

$$\begin{aligned}
\mu_{\delta_x}(S) &= \int \chi_S \, d\mu_{\delta_x} \\
&= \langle \delta_x, T_A(\chi_S)\delta_x \rangle \\
&= \langle P\delta_x, PT_A(\chi_S)\delta_x \rangle \\
&= \langle P\delta_x, T_A(\chi_S)P\delta_x \rangle \\
&= \langle \delta_y, T_A(\chi_S)\delta_y \rangle \\
&= \int \chi_S \, d\mu_{\delta_y} \\
&= \mu_{\delta_y}(S)
\end{aligned}$$

showing that $\mu_{\delta_x} = \mu_{\delta_y}$. □

The following corollary summarizes what we need to know about the relationship between spectral measures and the spectrum. We make a notation change at this point, and let e denote an arbitrary vertex in our vertex-transitive graph, freeing up the symbol x, which will have other jobs to do in the next section.

Corollary 8 *Let Γ be a vertex-transitive graph with adjacency operator A. Let e be a vertex in Γ, and let μ_{δ_e} be the spectral measure associated with δ_e. Then*

$$\mathrm{Spec}(A) = \overline{\mathrm{supp}(\mu_{\delta_e})}.$$

6 The search for Spec

We now focus our attention on the spectral measure μ_{δ_e}, which we will henceforth call simply μ. We'd like to describe this measure as completely as possible. We first decompose μ as

$$\mu \quad = \quad \mu_{\mathrm{ac}} + \mu_{\mathrm{sing}}$$

into parts which are absolutely continuous and singular, respectively, with respect to Lebesgue measure.

Since μ_{ac} is absolutely continuous with respect to Lebesgue measure, there is a function g, called the Radon-Nikodym derivative of μ_{ac}, satisfying

$$\int f \, d\mu_{\mathrm{ac}} \quad = \quad \int f g \, dx$$

for every Borel function f.

A clever procedure, known as the formula of Stieltjes, Stone, Kodaira, and Titchmarsh (see [8]), allows us to write down a formula for g in terms of the adjacency operator A, and ultimately in terms of the return generating function on Γ. We begin with the bump function

$$b_\lambda(x) \quad = \quad \frac{1}{2\pi i}\left(\frac{1}{x-\lambda-i\varepsilon} - \frac{1}{x-\lambda+i\varepsilon}\right).$$

It is easy to see (by finding a common denominator, for example) that b_λ is real-valued, and for ε small, $b_\lambda(x)$ is close to zero except when $x \approx \lambda$. A straightforward calculus exercise shows that $\int b_\lambda(x)\,dx = 1$, provided $\varepsilon > 0$. Using these two observations, one can prove (see [8] or [4])

Theorem 9 *If the function g is continuous at the point λ, then*

$$\lim_{\varepsilon \to 0^+} \int b_\lambda(x) g(x)\,dx \quad = \quad g(\lambda).$$

In particular, if g is the Radon-Nikodym derivative of μ_{ac}, then for each point λ at which g is continuous, we have

$$g(\lambda) \quad = \quad \lim_{\varepsilon \to 0^+} \int b_\lambda(x)\,d\mu_{\mathrm{ac}}.$$

Next, by the spectral theorem, this integral of a Borel function against the measure $d\mu_{\mathrm{ac}}$ can be expressed in terms of the operator A. We get

$$\begin{aligned}\int b_\lambda(x)\,d\mu_{\rm ac} &= \left\langle \delta_e, \frac{1}{2\pi i}\left([A-(\lambda+i\varepsilon)I]^{-1} - [A-(\lambda-i\varepsilon)I]^{-1}\right)\delta_e \right\rangle \\ &= \frac{1}{2\pi i}\left(\left\langle \delta_e, [A-(\lambda-i\varepsilon)I]^{-1}\delta_e\right\rangle - \left\langle \delta_e, [A-(\lambda+i\varepsilon)I]^{-1}\delta_e\right\rangle\right)\end{aligned}$$

In order to simplify the notation, we introduce a Green's function

$$G(w) \;=\; \left\langle \delta_e, (A-\overline{w}I)^{-1}\delta_e \right\rangle,$$

which is analytic everywhere except on Spec(A). This allows us to rewrite the equation above as

$$\int b_\lambda(x)\,d\mu_{\rm ac} \;=\; \frac{1}{2\pi i}[G(\lambda+i\varepsilon) - G(\lambda-i\varepsilon)].$$

Using Theorem 9, it follows that the derivative g of $\mu_{\rm ac}$ satisfies

$$g(\lambda) \;=\; \frac{1}{2\pi i}\lim_{\varepsilon\to 0^+}[G(\lambda+i\varepsilon) - G(\lambda-i\varepsilon)] \tag{7}$$

at every point λ where g is continuous.

We find an analytic expression for the function G by observing that it is related, at least formally, to the return generating function $R(z)$ of section 3. We have

$$\begin{aligned}G(w) &= \left\langle \delta_e, -\overline{w}^{-1}\left(I - \frac{1}{\overline{w}}A\right)^{-1}\delta_e \right\rangle \\ &= -\frac{1}{w}\left\langle \delta_e, \sum_{n=0}^{\infty}\left(\frac{1}{\overline{w}^n}A^n\right)\delta_e \right\rangle \\ &= -\frac{1}{w}\sum_{n=0}^{\infty}\langle \delta_e, A^n\delta_e\rangle \frac{1}{w^n} \\ &= -\frac{1}{w}R\left(\frac{1}{w}\right).\end{aligned}$$

Because $R(z)$ has a positive radius of convergence, the relationship above is more than formal; for all sufficiently large w, the function $G(w)$ must be equal to the function $-w^{-1}R(w^{-1})$. Since we have a closed-form expression (5) for $R(z)$, valid within its disk of convergence, we can write down a closed-form expression for $G(w)$, valid for all w outside some disk centered at 0. We get

$$G(w) = \frac{2(w^2-w)}{1+w-w^2+(1-2w)\sqrt{1+6w-5w^2-2w^3+w^4}}, \qquad (8)$$

where we have chosen the sign of the square root so that $G(w)$ agrees with $-w^{-1}R(w^{-1})$ for all real w where $R(w^{-1})$ converges. To simplify notation, let

$$\begin{aligned} A(w) &= 2(w^2-w) \\ B(w) &= 1+w-w^2 \\ C(w) &= 1-2w \\ D(w) &= 1+6w-5w^2-2w^3+w^4 \end{aligned}$$

and write

$$G(w) = \frac{A(w)}{B(w)+C(w)\sqrt{D(w)}}. \qquad (9)$$

Now $G(w)$ is analytic everywhere off $\mathrm{Spec}(A)$, and $\mathrm{Spec}(A)$ is a compact subset of the real line. From (9), $G(w)$ can have branch points only at the roots of the quartic $D(w)$. There are four such roots, at

$$w = \frac{1 \pm \sqrt{13 \pm 8\sqrt{2}}}{2}.$$

We denote these roots as r_1, r_2, r_3, and r_4, with $r_1 < r_2 < r_3 < r_4$. Topological considerations suggest that the "cut" along which $G(w)$ will fail to be analytic is the union of the two intervals $[r_1, r_2] \cup [r_3, r_4]$. This is also the subset of $\mathbb{R}$ in which $D(w) \le 0$.

We next produce a branch of $\sqrt{D(w)}$ which is compatible with this cut. We write $D(w) = (w-r_1)(w-r_2)(w-r_3)(w-r_4)$, and for each factor $(w-r_i)$, take the square root

$$\sqrt{w-r_i} \;=\; |w-r_i|^{\frac{1}{2}} e^{\frac{i}{2}\arg(w-r_i)}$$

where $-\pi < \arg(w-r_i) \le \pi$. Then the function

$$\sqrt{D(w)} \;=\; \prod_i \sqrt{w-r_i} \tag{10}$$

is continuous across the real axis except on the union of intervals $[r_1, r_2] \cup [r_3, r_4]$.

We are now ready to compute the derivative g using formula (7) and our analytic expression for the Green's function G.

Let $\lambda \in \mathbb{R}$. If $D(\lambda) > 0$ and $B(\lambda) + C(\lambda)\sqrt{D(\lambda)} \neq 0$ then G is continuous in a neighborhood of λ in the complex plane, so that

$$\frac{1}{2\pi i} \lim_{\varepsilon \to 0^+} [G(\lambda + i\varepsilon) - G(\lambda - i\varepsilon)] \;=\; 0,$$

so μ_{ac} does not see any part of the real line where $D(\lambda) > 0$ and $B(\lambda) + C(\lambda)\sqrt{D(\lambda)} \neq 0$.

The only zeroes of $B(\lambda) + C(\lambda)\sqrt{D(\lambda)}$ lie at $\lambda = -2$ and $\lambda = 0$. These points belong to the singular part of the spectrum of A; we will consider them a little later on when we discuss μ_{sing}. For now, we will simply ask μ_{ac} to ignore the points $\lambda = -2$ and $\lambda = 0$.

Thus the support of μ_{ac} is contained in the intervals $[r_1, r_2] \cup [r_3, r_4]$, where $D(\lambda) \le 0$. By our choice (10) of the square root, we find that

$$\lim_{\varepsilon \to 0^+} \sqrt{D(\lambda + i\varepsilon)} \;=\; \begin{cases} -i\sqrt{-D(\lambda)} & \text{if} \quad \lambda \in [r_1, r_2] \\ i\sqrt{-D(\lambda)} & \text{if} \quad \lambda \in [r_3, r_4] \end{cases}$$

and that

$$\lim_{\varepsilon \to 0^+} \sqrt{D(\lambda - i\varepsilon)} \;=\; \begin{cases} i\sqrt{-D(\lambda)} & \text{if} \quad \lambda \in [r_1, r_2] \\ -i\sqrt{-D(\lambda)} & \text{if} \quad \lambda \in [r_3, r_4]. \end{cases}$$

Using these limits, we determine that $\dfrac{1}{2\pi i} \lim_{\varepsilon \to 0^+} [G(\lambda + i\varepsilon) - G(\lambda - i\varepsilon)]$ is equal to

$$\frac{A(\lambda)C(\lambda)\sqrt{-D(\lambda)}}{\pi(B^2(\lambda) - C^2(\lambda)D(\lambda))} \quad \text{if} \quad \lambda \in [r_1, r_2]$$

$$-\frac{A(\lambda)C(\lambda)\sqrt{-D(\lambda)}}{\pi(B^2(\lambda)-C^2(\lambda)D(\lambda))} \quad \text{if} \quad \lambda \in [r_3, r_4]$$

Having found this expression, we use a computer to plot $g(\lambda)$. Figure 2 gives us a look at the part of the spectral measure for the adjacency operator on Γ which is continuous with respect to Lebesgue measure.

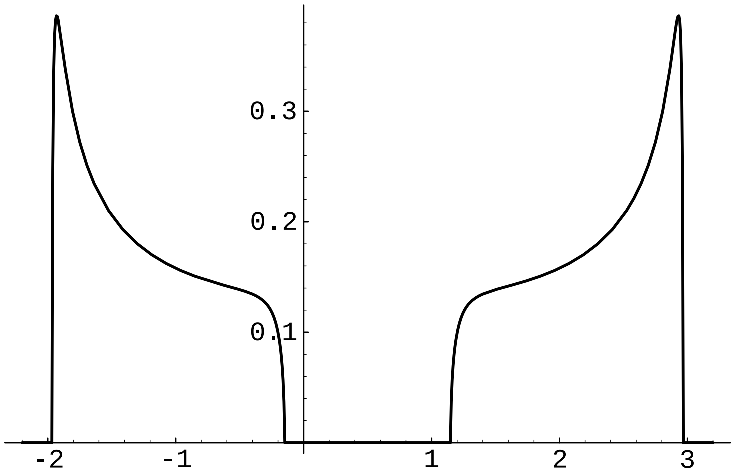

Figure 2: The Radon-Nikodym derivative of μ_{ac}

The continuous part of the spectrum of A is the closure of the support of μ_{ac}, that is, the union of intervals $[r_1, r_2] \cup [r_3, r_4]$.

Using a computer to integrate numerically, we find that the area under the curve in Figure 2 is about $\frac{2}{3}$, so μ_{ac} accounts for just that much of the spectrum of A. To see that the remaining $\frac{1}{3}$ of the spectrum lives in μ_{sing}, we hypothesize for the moment that μ_{sing} is concentrated at -2 and 0, the poles of $G(w)$. If so, then for any function f on $\mathbb{R}$, we have

$$\int f d\mu_{\mathrm{sing}} = c_1 f(-2) + c_2 f(0)$$

for some positive constants c_1 and c_2. In particular, if we take $f_\lambda(x) = (x-\lambda)^{-1}$ for $\lambda \in \mathbb{R}$, then by the functional calculus outlined in Theorem 5,

$$\begin{aligned}\int f_\lambda d\mu_{\mathrm{sing}} + \int f_\lambda d\mu_{\mathrm{ac}} &= \langle \delta_e, (A-\lambda I)\delta_e \rangle \\ &= G(\lambda).\end{aligned}$$

Thus

$$G(\lambda) \;=\; \int f_\lambda d\mu_{\rm ac} + \frac{c_1}{-2-\lambda} + \frac{c_2}{0-\lambda} \tag{11}$$

for some constants c_1 and c_2. In fact, since the numbers -2 and 0 lie outside the support of $\mu_{\rm ac}$, equation (11) says that c_1 and c_2 are minus the residue of $G(\lambda)$ at $\lambda = -2$ and $\lambda = 0$ respectively. We find that the residue of $G(\lambda)$ at both -2 and 0 is equal to $-\frac{1}{6}$, so $\mu_{\rm sing}$ assigns weight $\frac{1}{6}$ to each of these points.

Since $\int d(\mu_{\rm ac} + \mu_{\rm sing})$ adds up to 1, it appears that we have accounted for our entire spectral measure μ, and we have a complete spectral description of the operator A.

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