Knot labellings and knots without labellings

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1 Introduction

From two recent introductory books on knot theory ([1] and [7]), we learned about using labellings of knot diagrams to distinguish knots. For each prime $p \geq 3$, a diagram of a knot K has a valid mod-p labelling if and only if every diagram representing the knot type of K has a valid mod-p labelling. Writing down the first few examples that come to mind shows quickly and easily that the trefoil and the figure-eight knots are distinct from one another, and that neither of them is equivalent to the trivial knot. (These facts are in close agreement with all of our experimental data.)

We then wondered if there were any knots other than the trivial knot which fail to have mod-p labellings for any prime p. We learned somewhat later that knot theorists have long known that there are infinitely many such knots. In [4], it is shown that a diagram of a knot K has a mod-p labelling if and only if $\Delta_K(-1)$ is divisible by p, where $\Delta_K(t)$ is the Alexander polynomial of K. When K is an odd torus knot, its Alexander polynomial $\Delta_K(t)$ is readily computed (see [6], p. 265), and one finds that $\Delta_K(-1) = 1$ for every such knot. It can be shown by other methods (see [10], p. 53, for example) that there are infinitely many distinct odd torus knots, so we get an infinite family of knots, none of which has a mod-p labelling for any prime p.

This approach provides a swift and decisive answer to our question, but relies on rather advanced techniques. Since we are far from being specialists in the field of knot theory, we preferred to investigate mod-p labellings from a more elementary point of view. In doing so, we found that the study of knots provides some nice, hand-on applications of familiar techniques and results from group presentations, group representations, and number theory, as well as basic algebraic topology. Once we understood the algebraic significance of knot labellings (Lemma 5), we were able to produce a very accessible demonstration that the odd torus knots have no mod-p labellings.

2 Definitions

A *knot* is a mathematical model of a piece of rope which has had a knot tied in it and then had its ends spliced together. Thus a knot is basically an embedding of a circle in $\mathbb{R}^3$. Since a physical piece of rope has some positive thickness, a physical knot can't be pulled infinitely tight. This property is usually included in the definition of a mathematical knot by requiring that the embedded circle not come "arbitrarily close to itself." Embeddings of the circle in $\mathbb{R}^3$ which satisfy a condition of this kind are called *tame knots*. Knots that are not tame are called *wild*. Wild knots are not good models of everyday physical knots, and it seems they are not studied very much. In this article, when we say "knot," we will mean "tame knot."

We should consider two knots as equivalent if the ropes they model can be pushed around in $\mathbb{R}^3$ so that they look the same, at least up to scaling. Pushing a rope around in space is modelled by the mathematical notion of an *ambient isotopy*, which is a family of homeomorphisms $f_t : \mathbb{R}^3 \to \mathbb{R}^3$ where the "time" variable t runs through the interval $[0, 1]$. The initial homeomorphism f_0 is the identity, and the family f_t, considered as a function from $\mathbb{R}^3 \times [0, 1]$ to $\mathbb{R}^3$, must be continuous. An ambient isotopy is like a movie: for each $t \in [0, 1]$, the map f_t corresponds to a single frame showing the position of our rope at time t. If $K \subseteq \mathbb{R}^3$ is a knot and f_t is an ambient isotopy such that $f_t(K)$ is a knot for each $t \in [0, 1]$ (that is, the rope doesn't pass through itself at any time during the movie), then the knot $f_1(K)$ is *isotopy equivalent* to K. It is intuitively clear (and easy enough to check formally) that isotopy equivalence is indeed an equivalence relation. When two knots are isotopy equivalent, we will say that they are of the same *type*.

To study knots, we draw pictures of them. A *knot diagram* of a knot K is a projection K onto a plane, with gaps in the curve to indicate where parts of K cross under other parts of K. We also insist that each point in the projection be the image of no more than two points in K, and that all intersections in the diagram be transverse.

A *knot invariant* is some quantity we assign to a knot K that depends only on the knot type of K, and not on the particular representative we happen to be looking at. Knot invariants that we can read off a knot diagram are particularly useful, because they can tell us immediately that two diagrams represent different knot types.

3 Knot labellings

A *strand* in a knot diagram is an arc between one undercrossing and the next. If a knot diagram has n crossings, then it has n strands. A *labelling* of a knot diagram is a mapping from the strands of the diagram into some set of symbols. If we number the strands (in some arbitrary way) as $1, 2, \ldots, n$, then we can write down any labelling as an n-tuple $(a_1, a_2, \ldots, a_n)$, indicating that the symbol a_i is assigned to strand i.

Let p be a prime. A *mod-p labelling* of a knot diagram is a labelling using the symbols $\{0, 1, \ldots, p-1\}$. A mod-p labelling $(a_1, a_2, \ldots, a_n)$ is called *valid* if it satisfies the two conditions

MPL1 At least two different labels are used.

MPL2 At each crossing the relation $a_j + a_k - 2a_i \equiv 0 \pmod p$ holds, where i is the overcrossing strand, and j and k are the strands that form the undercrossing.

Claim 1 *The existence of a valid mod-p labelling is a knot invariant. That is, one diagram of a knot K has a valid mod-p labelling if and only if every diagram of every knot of the same type as K has a valid mod-p labelling.*

This is shown in [7] by combinatorial methods. We will outline a slightly different proof (also suggested in [7]) later on, but for now, let's accept Claim 1 and try it out on some familiar knots. In Figure 1, it is easy to check that the mod-3 labelling of the trefoil and the mod-5 labelling of the figure-eight knot are both valid. It is just a shade more difficult to check that the trefoil diagram can have no valid mod-5 labelling. (Hint: If (a_1, a_2, a_3) is a valid mod-p labelling, then so is $(a_1 + c, a_2 + c, a_3 + c)$, reading the entries mod p, for any constant c.) By Claim 1, the two diagrams in Figure 1 actually represent distinct knots. Furthermore, since the trivial knot (that is, an unknotted loop of rope) has a diagram with only one strand, and no labelling of such a diagram can satisfy MPL1, the trivial knot has no mod-p labelling for any p. Thus neither the trefoil nor the figure-eight knot is equivalent to the trivial knot.

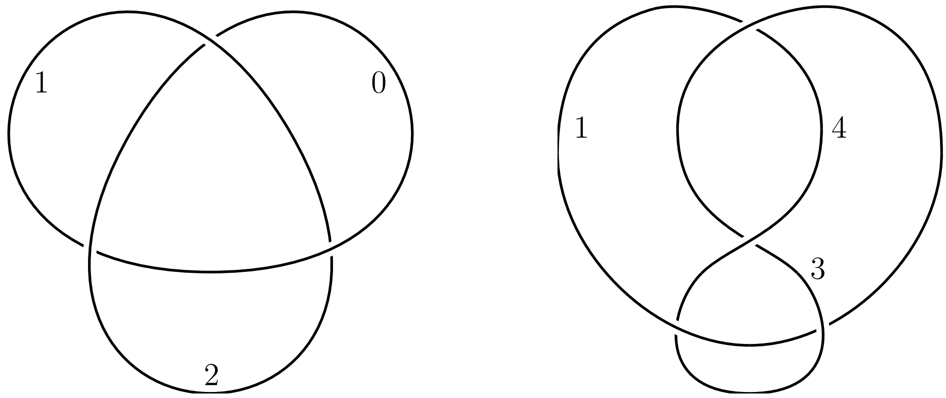

Figure 1: A valid mod-3 labelling of a trefoil diagram (left) and a valid mod-5 labelling of a figure-eight diagram (right).

For our next labelling, it will be convenient to give our knot diagram an orientation, at least temporarily. This is easy to do. We orient a knot by choosing a direction in which to traverse it. The knot orientation is inherited by the diagram, and may be indicated by drawing arrowheads along the strands. A crossing in an oriented knot diagram is *right-handed* if an observer walking along the overcrossing strand in the preferred direction sees the undercrossing traffic approaching from the right. Otherwise the crossing is *left-handed*. (See figure 2.) The handedness of a crossing does not depend on the orientation chosen for the knot; reversing all the arrowheads takes a right-handed crossing to another right-handed crossing.

Figure 2: A right-handed crossing (left) and a left-handed crossing (right).

Let H be a group and K be a knot. If we have an oriented diagram of K with n strands numbered $1, 2, \ldots, n$, we can form an *H-labelling* of the diagram by assigning a group element h_i to each strand i. An H-labelling $(h_1, h_2, \ldots, h_n)$ is called *valid* if the following two conditions are satisfied.

GL1 The set $\{h_1, h_2, \ldots, h_n\}$ is a generating set for H.

GL2 At each right-handed crossing, the relation $h_i h_k h_i^{-1} = h_j$ holds, where i is the overcrossing strand, j is the "incoming" undercrossing strand, and k is the "outgoing" undercrossing strand. At each left-handed crossing, with i, j, and k in the same roles, the relation $h_i h_j h_i^{-1} = h_k$ holds.

Since GL2 refers to "incoming" and "outgoing" strands, it appears that the validity of an H-labelling depends on the orientation we choose for our knot diagram. Although this is so, the *existence* of a valid H-labelling is independent of our choice of orientation. This is because $(h_1, h_2, \ldots, h_n)$ is a valid H-labelling of a diagram M if and only if $(h_1^{-1}, h_2^{-1}, \ldots, h_n^{-1})$ is a valid H-labelling of the diagram M' obtained from M by reversing its orientation.

As we did with mod-p labellings, we now assert that group labellings are useful, but we defer giving the proof until later on.

Claim 2 *The existence of a valid H-labelling is a knot invariant.*

As we travel around a knot diagram with a valid H-labelling, the condition GL2 tells us that the label on each new strand we encounter (beginning at some undercrossing) is conjugate to the label on the preceding strand. It follows that all the labels in a valid H-labelling of a knot diagram must belong to a single conjugacy class in H. Since, by GL1, these labels must generate H, any group we use to label a knot diagram must be generated by the elements of a single conjugacy class. In particular, no non-trivial abelian group can be used in a valid group labelling of a knot.

When we turn our attention to the dihedral groups, though, we find just what we're looking for. Let $p \geq 3$ be a prime, and let D_p denote the dihedral group of order $2p$, which we present as

$$D_p \quad = \quad \left\langle r, s \,:\, r^p = s^2 = 1,\, rs = sr^{-1} \right\rangle .$$

We list all the elements of D_p as

$$\{1, r, r^2, \ldots, r^{p-1}, s, sr, sr^2, \ldots sr^{p-1}\},$$

and consider this group's conjugacy classes. Each element r^k is conjugate only to its inverse, but the conjugacy class containing s also contains sr^k for each k. (For even k, conjugate s by $r^{\frac{k}{2}}$; for odd k, conjugate s by $r^{\frac{p-k}{2}}$.)

Can elements of the conjugacy class of s generate D_p? Since each sr^k has order 2, we will need at least two such elements to have any hope of generating D_p. So consider sr^k and sr^l with $k \not\equiv l \pmod{p}$, and let H be the subgroup of D_p generated by these two elements. Then H contains $sr^k sr^l = r^{l-k}$. Since p is prime, $l-k$ is relatively prime to p, and elementary number theory tells us that some power of r^{l-k} is equal to r in D_p. So H contains r. Since H also contains sr^k, it follows that H contains s. Thus $H = D_p$.

We summarize this discussion as a lemma.

Lemma 3 *Let $p \geq 3$ be a prime. If a set of elements from a single conjugacy class of D_p generates D_p, the elements must be of the form sr^k. Furthermore, any two distinct elements of the form sr^k generate D_p.*

The condition GL1 says that a valid D_p-labelling must use only labels of the form sr^k, and it must use at least two of them. This echo of the condition MPL1 suggests that mod-p labellings and D_p-labellings are somehow related. In fact, a closer look shows us that mod-p labellings and D_p-labellings are actually identical.

Lemma 4 *Let $p \geq 3$ be a prime and M a knot diagram. Then M has a valid mod-p labelling if and only if M has a valid D_p-labelling.*

Proof Suppose M has n strands. By the discussion above, any valid D_p-labelling of M must be of the form $(sr^{a_1}, sr^{a_2}, \ldots, sr^{a_n})$, where each a_i is in the set $\{0, 1, \ldots, p-1\}$. We claim that the D_p-labelling $(sr^{a_1}, sr^{a_2}, \ldots, sr^{a_n})$ is valid if and only if the mod-p labelling $(a_1, a_2, \ldots, a_n)$ is valid.

First, GL1 and MPL1 are equivalent in this situation, because each one simply requires the appearance of two distinct a_i.

To see that GL2 is equivalent to MPL2, consider a right-handed crossing with the usual cast of characters: i is the overcrossing strand and j and k are, respectively, the incoming and outgoing undercrossing strands. In the D_p labelling, the condition GL2 at this crossing says that

$$\begin{aligned} 1 &= (sr^{a_i})(sr^{a_k})(sr^{a_i})^{-1}(sr^{a_j})^{-1} \\ &= r^{a_j+a_k-2a_i}, \end{aligned}$$

that is, $a_j + a_k - 2a_i$ is a multiple of p. But this is exactly condition MPL2.

Interchanging all the j's and k's in the previous paragraph shows that GL2 and MPL2 are equivalent at left-handed crossings, as well. $\square$

We remark that, although we still have not proved either Claim 1 or Claim 2, Lemma 4 at least shows that Claim 2 implies Claim 1.

4 Knot groups

The topology of a knot isn't useful for distinguishing one knot from another, since every knot is homeomorphic to a circle. Knot theorists study instead the topology of the complement of a knot, and in particular the fundamental group $\pi_1(\mathbb{R}^3 - K)$, which is called the *knot group* of K. We will denote this group $G(K)$. Since the topology of $\mathbb{R}^3 - K$ is preserved by ambient isotopies, any quantity depending on $G(K)$ is automatically a knot invariant.

An element of $G(K)$ is represented by an oriented closed path which begins at some fixed base point $x_0 \notin K$, winds through the space around K and then returns to x_0. The composition operation in this group corresponds to concatenation of paths. The identity element is represented by a path that never leaves x_0, or by a loop at x_0 which never gets tangled up with any part of K, so that it can be shrunk back to x_0 without getting caught anywhere. The inverse of the group element represented by a path σ is represented by the same path traced in the opposite direction.

In Figure 3, for example, if path σ_1 represents a group element g, then path σ_2 represents g^{-2}. Path σ_3 represents the identity, since it can be pulled clear of the knot.

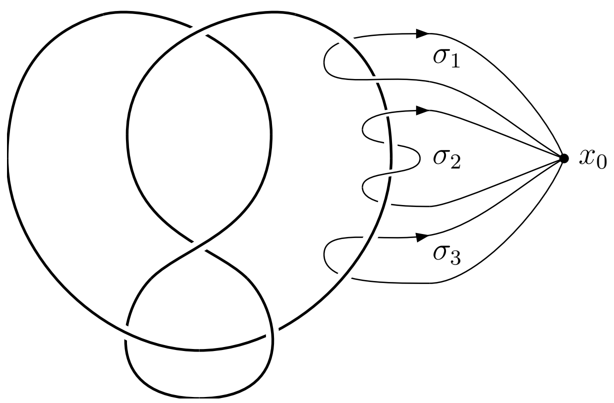

Figure 3: Three paths in the complement of a figure-eight knot.

Given an oriented diagram of a knot K, there is a nice recipe for writing down a group presentation for $G(K)$. We first establish a base point x_0 somewhere off to the side of our knot diagram. Then for each strand i in the diagram, we write down a group element g_i, represented by a closed path which begins at x_0, crosses under strand i from, say, right to left, then crosses over strand i and returns to x_0 without getting tangled up anywhere else in the knot. (See Figure 4.) It is not too surprising that the elements g_i actually generate $G(K)$. This is just saying that any x_0-based loop through the space around K can be deformed into a sequence of loops each of which leaves x_0, circles one strand of K (in one direction or the other), and then returns to x_0.

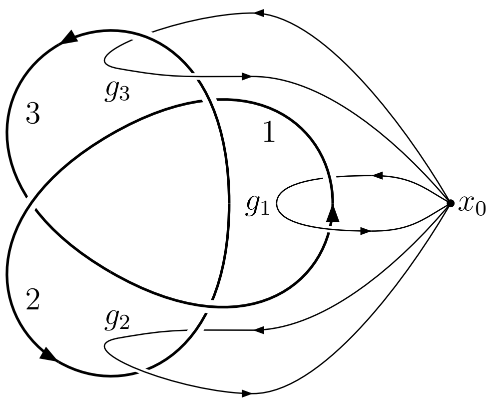

Figure 4: Representatives of the generators g_1, g_2, and g_3 of the knot group of a trefoil knot.

For each crossing in the knot diagram, we have a relation among the generators g_i. At a right-handed crossing where i is the overcrossing strand and j and k are the incoming and outgoing undercrossing strands, a path which passes underneath all three of these strands and circles the crossing once (see Figure 5) represents the group element $g_i g_k g_i^{-1} g_j^{-1}$, since it can be deformed into four paths representing these generators. On the other hand, since this path can be pulled clear of K, it represents the identity in $G(K)$, so we get the relation $g_i g_k g_i^{-1} g_j^{-1} = 1$. Hold Figure 5 up to a mirror to see that the corresponding relation at a left-handed crossing is $g_i^{-1} g_k^{-1} g_i g_j = 1$.

We now have a set of generators (one for each strand) and a set of relations (one for each crossing). The wonderfully useful result is that this set of relations is sufficient to define $G(K)$. The recipe we just outlined is called the *Wirtinger presentation* of $G(K)$. A proof that this presentation actually does describe the knot group of K is found in [10].

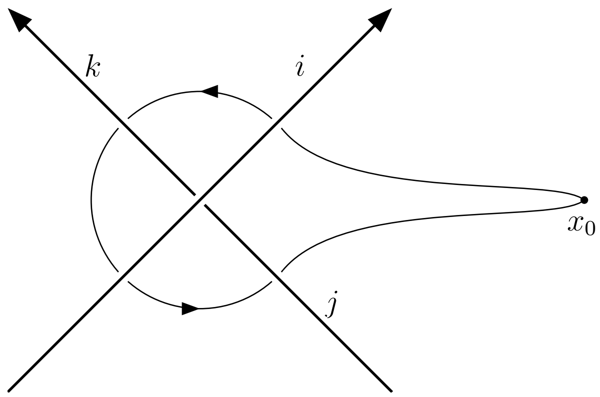

Figure 5: At a right-handed crossing, the generators of a knot group satisfy the relation $g_i g_k g_i^{-1} g_j^{-1} = 1$.

5 Knot groups and knot labellings

Using the Wirtinger presentation, we can establish the important connection between knot groups and group labellings of knots. This connection will put the knot labellings, which we described in a combinatorial way, on a more topological footing. It will also allow us to use algebraic techniques to study the labellings themselves, and in certain cases, to show that valid labellings cannot be constructed.

Lemma 5 *Let K be a knot and H a group.*

1. *If any oriented diagram of K has a valid H-labelling, then there exists a surjective homomorphism from $G(K)$ to H.*
2. *If there exists a surjective homomorphism from $G(K)$ to H, then every oriented diagram of K has a valid H-labelling.*

We sketch the proof.

For the first statement, suppose we have a diagram of K with a valid H-labelling. Then for each i, strand i in the diagram is associated with some generator g_i of $G(K)$ and some label h_i from H. The natural candidate for a homomorphism from $G(K)$ to H is a map taking each g_i to the corresponding h_i. It turns out that each Wirtinger relation among the g_i corresponds to a relation (required by GL2) among the h_i, so that the map taking g_i to h_i can indeed be extended to a homomorphism. (To show this formally, one writes down presentations for the two groups and applies Van Dyck's Theorem (see [5]).) The condition GL1 guarantees that our homomorphism is surjective.

For the second statement, we are given a surjective homomorphism $\varphi : G(K) \to H$ and a diagram of K. The natural way to label the diagram is to assign to each strand i the group element $\varphi(g_i)$, where g_i is the strand's Wirtinger generator. The Wirtinger relations map to just the relations required by GL2, and the fact that φ is surjective is exactly GL1. □

Claim 2 now follows easily from Lemma 5 and, in turn, establishes Claim 1.

The idea of labelling knot diagrams seems to have originated with R. H. Fox. In [4], he explains how a knot group may be studied by considering its homomorphic images in metacyclic groups. Specializing to the dihedral groups leads directly to the mod-p labellings which were our starting point.

6 Torus knots

Let m and n be relatively prime. An (m, n) *torus knot*, denoted $T_{m,n}$, is a simple closed curve that winds around a standard torus m times in the longitudinal direction and n times in the meridional direction. More concretely, if we let $\rho : \mathbb{R}^2 \to \mathbb{R}^3$ by

$$\rho(x, y) = ((2 + \cos 2\pi y) \cos 2\pi x, (2 + \cos 2\pi y) \sin 2\pi x, \sin 2\pi y),$$

then the image under ρ of the line segment from the origin to the point (m, n) is an (m, n) torus knot. Figure 6 shows the torus knot $T_{3,5}$ on the surface of a standard torus.

It is known ([9], Proposition 7.5) that the number of crossings in any diagram of an (m, n) torus knot is at least the minimum of $m(n-1)$ and $n(m-1)$. For larger values of m and n, this makes the Wirtinger presentation of $G(T_{m,n})$ unwieldy. However, it is easy to compute the knot group of a torus knot using the Seifert-Van Kampen theorem. In fact, finding $G(T_{m,n})$ is literally a textbook example (see [8], p. 136 or [3], p. 92) of a Seifert-Van Kampen computation.

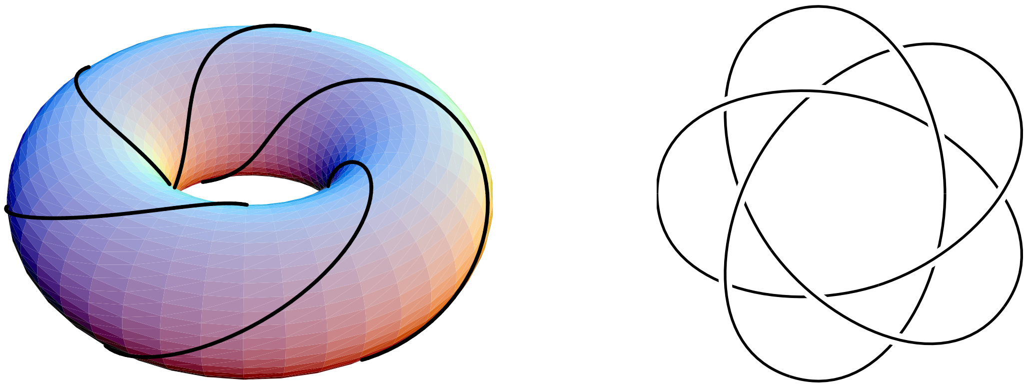

Figure 6: The torus knot $T_{3,5}$ on the surface of a torus, and a ten-strand diagram of $T_{3,5}$.

Recall that the Seifert-Van Kampen theorem describes the fundamental group of a path-connected union $X_1 \cup X_2$ in terms of the fundamental groups of X_1, X_2, and $X_1 \cap X_2$. Roughly speaking, we obtain a presentation of $\pi_1(X_1 \cup X_2)$ by combining the generators and relations from $\pi_1(X_1)$ and $\pi_1(X_2)$, and then throwing in an extra relation for each generator of $\pi_1(X_1 \cap X_2)$. The extra relations account for the fact that a closed path σ in $X_1 \cap X_2$ represents simultaneously some element of $\pi_1(X_1)$ and some element of $\pi_1(X_2)$. In $\pi_1(X)$, the two group elements represented by σ must be equal, and so we include a relation that says so.

To compute the fundamental group of $\mathbb{R}^3 - T_{m,n}$, we place $T_{m,n}$ on a standard torus, as in Figure 6. The intersection of this torus with the complement of $T_{m,n}$ is a ribbon which winds around the torus m times in the longitudinal direction and n times in the meridional direction. Let R denote this ribbon. The space X_1 will be the region inside the torus, along with R, and the space X_2 will be the region outside the torus, along with R. We choose a point somewhere in $X_1 \cap X_2$ to serve as a base point for all spaces concerned.

The space X_1 is topologically a solid torus, so its fundamental group is a free group on a single generator x, corresponding to a path that makes one trip around the "hole" in the torus. It is not immediately obvious that the space X_2 is also topologically a solid torus, but it should be plausible that its fundamental group is also a free group on a single generator y, corresponding to a path that passes once through the hole.

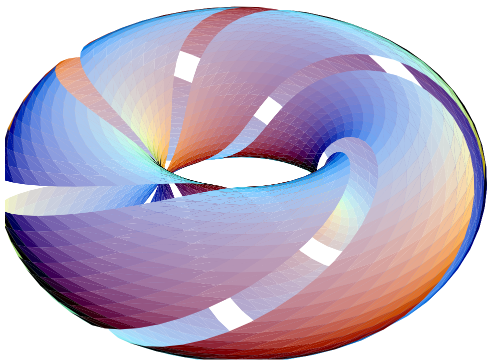

Figure 7: The space $R = X_1 \cap X_2$ in the complement of $T_{3,5}$.

The intersection of X_1 and X_2 is the ribbon R. Since R is topologically an annulus, its fundamental group is also free on one generator, which is represented by a path σ making one trip around the annulus. From the point of view of X_1, the path σ makes m trips around the torus, so it represents x^m. From X_2's point of view, σ passes n times through the hole in the torus, so it represents y^n.

The result of all this is that the knot group of $T_{m,n}$, which is the fundamental group of $X_1 \cup X_2$, has the simple presentation

$$G(T_{m,n}) \quad = \quad \langle x, y \, : \, x^m = y^n \rangle \, .$$

7 Knots with no mod-p labellings

An (m, n) torus knot is called *odd* if both m and n are odd. To finish our proof that odd torus knots have no mod-p labellings, we need to show that there can be no surjective homomorphism from the group $G(T_{m,n})$ to the group D_p when m and n are both odd and p is an odd prime.

To begin, we write down the group presentations $G(T_{m,n}) = \langle x, y \,:\, x^m = y^n \rangle$ and $D_p = \langle r, s \,:\, r^p = 1, s^2 = 1, sr = r^{-1}s \rangle$, and suppose that we have a homomorphism $\varphi : G(T_{m,n}) \to D_p$. Next, we apply Barry Commoner's second law of ecology (see [2]), "Everything must go somewhere." In the present instance, this says that $\varphi(x)$ must be either r^k for some k or else sr^k for some k, since these account for all the elements of D_p. Similarly, $\varphi(y)$ is either r^j or sr^j for some j. We consider the four possible cases.

		Image of y	
		r^j	sr^j
Image of x	r^k	I	II
	sr^k	III	IV

Case I Since $\varphi(x)$ and $\varphi(y)$ are both powers of r, the entire image of φ is contained in the cyclic subgroup of D_p generated by r. Thus φ is not surjective.

Case II Since φ is a homomorphism and $x^m = y^n$ in $G(T_{m,n})$, we must have

$$(\varphi(x))^m = \varphi(x^m) = \varphi(y^n) = (\varphi(y))^n. \tag{1}$$

The order of sr^j is 2 and n is odd, so $(\varphi(y))^n = (sr^j)^n = sr^j$. So sr^j must also be equal to $(\varphi(x))^m$, which in this case is r^{km}. But r^{km} is a power of r and sr^j is not, so this case cannot occur.

Case III This is just Case II with the roles of x and y interchanged.

Case IV Because m and n are odd and the elements sr^j and sr^k are of order 2, equation (1) implies that $sr^j = sr^k$. But if x and y both map to the same element and that element has order 2, then the image of φ contains only two elements, so it can't be all of D_p.

We have shown that there can be no surjective homomorphism from $G(T_{m,n})$ to D_p, and thus arrived by fairly elementary means at an answer to our question about labellings.

Theorem 6 *An odd torus knot has no valid mod-p labelling for any prime $p \geq 3$.*

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