Isospectrality conditions for regular graphs

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1 Introduction

Let $k \geq 3$ be an integer and let Γ be a k-regular, undirected graph with a finite number N of vertices. If A is an adjacency matrix for Γ, then the multiset $\{\lambda_1, \lambda_2, \ldots, \lambda_N\}$ of eigenvalues of A is called the *spectrum* of the graph Γ. Since Γ is undirected, A is symmetric and so its eigenvalues are all real. Also, since A is determined by Γ up to a permutation of the rows and columns, the spectrum of Γ does not depend on the particular adjacency matrix A.

The universal cover of any k-regular graph is the k-tree, which we will denote X_k. We let G denote the group of automorphisms of X_k. A k-regular graph Γ may then be viewed as the quotient of X_k by a freely-acting subgroup H of G. The vertices of $\Gamma = H\backslash X_k$ are the orbits Hx of vertices in X_k, and Hx is adjacent to Hy in $H\backslash X_k$ if and only if each element of Hx is adjacent to some element of Hy in X_k.

Our main result is

Theorem 12 *Let $H\backslash X_k$ be an N-vertex, k-regular graph with no loops or parallel edges. For each integer $n \geq 1$, let*

$$P_n = \sum_{[h_i]_H \subset [t_n]_G} L(C_H(h_i))$$

where $[t_n]_G$ is the G-conjugacy class containing all length-n translations in H and $L(C_H(h_i))$ denotes the length of a generator of the centralizer $C_H(h_i)$ of h_i in H.

Then the spectrum of $H\backslash X_k$ determines and is determined by the sequence $P_1, P_2, \ldots, P_N$.

We view this theorem as a combinatorial analogue to a theorem of DeTurck and Gordon ([2], Theorem 1.16). Their results states that two compact quotients $H_1\backslash M$ and $H_2\backslash M$ of a Riemannian manifold M by uniform discrete subgroups H_1 and H_2 of Lie group G acting freely and properly discontinuously by isometries on M are isospectral if, for each $h \in G$,

$$\sum_{[h_i]_{H_1} \subset [h]_G} \rho^{(h_i)}(C_{H_1}(h_i)\backslash C_G(h_i)) \quad = \quad \sum_{[h_i]_{H_2} \subset [h]_G} \rho^{(h_i)}(C_{H_2}(h_i)\backslash C_G(h_i))$$

where the $\rho^{(h_i)}$ are ad hoc Haar measures, brought briefly into the picture in order to measure the centralizers of elements h_i in H_1 and H_2. In the graph-theoretic setting, the centralizers $C_H(h_i)$ are cyclic, and the Haar measures are replaced by the notion of the length of a generator.

Both our Theorem 12 and the DeTurck and Gordon theorem are descendants of the isospectrality conditions introduced by Sunada ([7]) in 1985.

2 Definitions

We begin by defining some of our graph-theoretic terms.

Let Γ be a graph (always assumed to be undirected). An *n-walk* W in Γ is a sequence

$$(x_0, e_1, x_1, e_2, x_2, \ldots, e_n, x_n)$$

of vertices x_i and edges e_i such that each e_i is incident on x_{i-1} and x_i. We say the walk *begins* at the vertex x_0 and *ends* at the vertex x_n. An n-walk W is *closed* if $x_0 = x_n$. It is *non-backtracking* (denoted NBT) if $e_i \neq e_{i+1}$ for each i.

Let $W = (x_0, e_1, x_1, e_2, x_2, \ldots, e_n, x_n)$ be a closed NBT n-walk. If $e_1 \neq e_n$, then W is *tailless*. Otherwise, since W is NBT, there is a greatest integer

$r < n/2$ for which $e_r = e_{n-r}$ in W. We say that W has a *tail* of length r, and that

$$W' = (x_r, e_{r+1}, x_{r+1}, \ldots, e_{n-r-1}, x_{r-n-1})$$

is the *tailless part* of W.

There is a natural equivalence relation on tailless NBT n-walks in which two such n-walks

$$W_1 = (x_0, e_1, x_1, \ldots, e_n, x_n) \quad \text{and} \quad W_2 = (y_0, f_1, y_1, \ldots, f_n, y_n)$$

are equivalent if and only if the vertices and edges of W_2 are a cyclic permutation of those of W_1. That is, W_1 and W_2 are equivalent if and only if there is some j with $y_i = x_{i+j}$ and $f_i = e_{i+j}$ for all i, where the subscripts are read modulo n. The class of a tailless NBT n-walk under this equivalence is called the *cycle* represented by W, and will be denoted $\langle W \rangle$.

The k-tree X_k is simply connected, and thus given any two vertices x and y in X_k, there is a unique NBT n-walk beginning at x and ending at y. In this case, we define the *distance* between x and y to be n, and denote it by $\mathrm{dist}(x, y)$. It is easily verified that our definition yields an actual distance function.

A *geodesic* in X_k is an infinite NBT walk.

3 Graph-theoretic Bessel functions

We now turn to the k-tree analogue of Bessel functions, which help to make the connection between the spectral and combinatorial properties of a regular graph.

Let $V(X_k)$ denote the vertex set of X_k. Let $\mathcal{A}$ be the operator on the function space $\mathbb{C}^{V(X_k)}$ given by

$$(\mathcal{A}f)(x) \quad = \quad \sum_{y \sim x} f(x)$$

for $f \in \mathbb{C}^{V(X_k)}$. The notation "$y \sim x$" means that the vertices x and y are joined by an edge.

Let x_0 be a vertex in X_k and for each non-negative integer l, let $C(x_0, l)$ denote the set $\{x \in V(X_k) \,:\, \mathrm{dist}(x_0, x) = l\}$.

Lemma 1 *For each real number λ, there is a unique function $S_\lambda : \mathbb{N} \to \mathbb{C}$ such that if $f : V(X_k) \to \mathbb{C}$ is any function satisfying $\mathcal{A}f = \lambda f$, then*

$$\sum_{x \in C(x_0,l)} f(x) \;=\; f(x_0) S_\lambda(l)$$

for each non-negative integer l.

Furthermore, for each l, $S_\lambda(l)$ is a polynomial in λ of degree l.

Proof Without loss of generality, assume $f(x_0) \neq 0$. We proceed by induction. For $l = 0$, we have $C(x_0, 0) = \{x_0\}$, which implies that $S_\lambda(0) = 1$. For $l = 1$, the condition $\mathcal{A}f = \lambda f$ requires that

$$\lambda f(x_0) = \sum_{x \in C(x_0,1)} f(x) = f(x_0) S_\lambda(1)$$

so that $S_\lambda(1) = \lambda$. Now assume $l > 1$ and the statement holds for $l - 1$ and $l - 2$. Then the condition $\mathcal{A}f = \lambda f$ requires that

$$\sum_{x \in C(x_0,l-1)} \lambda f(x) \;=\; (k-1) \sum_{x \in C(x_0,l-2)} f(x) + \sum_{x \in C(x_0,r)} f(x).$$

Using the inductive hypothesis, we get

$$f(x_0)\lambda S_\lambda(l-1) \;=\; (k-1) f(x_0) S_\lambda(l-2) + f(x_0) S_\lambda(l).$$

This uniquely determines $S_\lambda(l)$ as $\lambda S_\lambda(l-1) - (k-1) S_\lambda(l-2)$. Again by the inductive hypothesis, $S_\lambda(l-1)$ and $S_\lambda(l-2)$ are polynomials in λ of degrees $l-1$ and $l-2$ respectively, so $S_\lambda(l)$ is a polynomial in λ of degree l. $\square$

Explicit formulas for these Bessel functions, along with some geometric applications, are found in [1] and [5].

4 Point-pair invariants

A function $f : V(X_k) \times V(X_k) \to \mathbb{C}$ is called a *point-pair invariant* if $f(x, y)$ depends only on the distance between x and y.

For each non-negative integer l, let $\delta_l : V(X_k) \times V(X_k) \to \mathbb{C}$ by

$$\delta_l(x, y) = \begin{cases} 1 & \text{if } \operatorname{dist}(x, y) = l \\ 0 & \text{otherwise.} \end{cases}$$

If $f : V(X_k) \times V(X_k) \to \mathbb{C}$ is a point-pair invariant, then f may be written

$$f(x, y) = \sum_{l=0}^{\infty} \pi_l \delta_l(x, y)$$

for some coefficients π_l.

Let $\Gamma = H \backslash X_k$ be a finite, k-regular graph and let f be a point-pair invariant on X_k. Let $F : V(\Gamma) \times V(\Gamma) \to \mathbb{C}$ by

$$F(x, y) = \sum_{h \in H} f(\tilde{x}, h\tilde{y}) \qquad (1)$$

where $\tilde{x}$ is a some lift of x and $\tilde{y}$ is some lift of y.

Lemma 2 *With the notation above,*

$$\operatorname{Tr} F = \sum_{i=1}^{N} \sum_{l=0}^{\infty} \pi_l S_{\lambda_i}(l)$$

where $\{\lambda_1, \lambda_2, \ldots, \lambda_N\}$ *is the spectrum of* Γ.

Proof (Also see [5].) Let A denote the operator on $\mathbb{C}^{V(\Gamma)}$ given by

$$A\varphi(x) = \sum_{y \sim x} \varphi(y)$$

For $\varphi \in \mathbb{C}^{V(\Gamma)}$. Then A may be represented by an adjacency matrix of Γ, so A is self-adjoint (since adjacency matrices for Γ are symmetric) and the set of eigenvalues of A is exactly the spectrum of Γ. Since A is self-adjoint, there exists an orthonormal set $\{\varphi_1, \varphi_2, \ldots, \varphi_N\}$ of eigenfunctions of A, corresponding to the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_N$ in the spectrum of Γ. Moreover, any lift of a λ-eigenfunction from Γ to X_k will be a λ-eigenfunction of the adjacency operator $\mathcal{A}$ on X_k.

Expanding the function $F(x,y)$ in terms of the φ_i, we get, for $x, y \in V(\Gamma)$,

$$F(x,y) = \sum_{i,j} a_{ij}\varphi_i(x)\varphi_j(y) \tag{2}$$

for some coefficients a_{ij}. We now calculate $\sum_{y\in V(\Gamma)} F(x,y)\varphi_s(y)$ in two different ways. First, using (2), we have

$$\sum_{y\in V(\Gamma)} F(x,y)\varphi_s(y) = \sum_{y\in V(\Gamma)} \sum_{i,j} a_{ij}\varphi_i(x)\varphi_j(y)\varphi_s(y) \tag{3}$$

$$= \sum_{i} a_{is}\varphi_i(x). \tag{4}$$

Second, letting $\tilde{x}$ and $\tilde{y}$ denote lifts of x and y respectively, and using $\tilde{\varphi}_s$ to denote a lift of the function φ_s, we have

$$\begin{aligned}\sum_{y\in V(\Gamma)} F(x,y)\varphi_s(y) &= \sum_{y\in V(\Gamma)} \sum_{h\in H} f(\tilde{x}, h\tilde{y})\varphi_s(y) \\ &= \sum_{v\in V(X_k)} f(\tilde{x}, v)\tilde{\varphi}_s(v) \\ &= \sum_{v\in V(X_k)} \sum_{l=0}^{\infty} \pi_l \delta_l(\tilde{x}, v)\tilde{\varphi}_s(v) \\ &= \sum_{l=0}^{\infty} \pi_l \sum_{v\in C(\tilde{x},l)} \tilde{\varphi}_s(v).\end{aligned}$$

Now $\tilde{\varphi}_s$ is a λ_s eigenfunction of $\mathcal{A}$, so by Lemma 1, we have for each l

$$\begin{aligned}\sum_{v\in C(\tilde{x},l)} \tilde{\varphi}_s(v) &= \tilde{\varphi}_s(\tilde{x})S_{\lambda_s}(l) \\ &= \varphi_s(x)S_{\lambda_s}(l).\end{aligned}$$

Thus,

$$\sum_{y \in V(\Gamma)} F(x,y)\varphi_s(y) \;=\; \varphi_s(x) \sum_{l=0}^{\infty} \pi_l S_{\lambda_s}(l). \tag{5}$$

From lines (4) and (5) we have

$$\sum_i a_{is}\varphi_s(x) \;=\; \varphi_s(x) \sum_{l=0}^{\infty} \pi_l S_{\lambda_s}(l),$$

from which we conclude that $a_{is} = 0$ if $i \neq s$ and

$$a_{ii} \;=\; \sum_{l=0}^{\infty} \pi_l S_{\lambda_i}(l).$$

Thus

$$F(x,y) \;=\; \sum_{i=1}^{N} \sum_{l=0}^{\infty} \pi_l S_{\lambda_i}(l)\varphi_i(x)\varphi_i(y)$$

from which the result follows. □

Corollary 3 *The trace of any function on $V(\Gamma) \times V(\Gamma)$ which arises from a point-pair invariant via formula (1) is determined by the spectrum of Γ.*

5 Closed NBT walks and the spectrum

Let $\Gamma = H\backslash X_k$ be a finite k-regular graph with N vertices. For each integer $l \geq 0$, let $D_l : V(\Gamma) \times V(\Gamma) \to \mathbb{C}$ by

$$D_l(x,y) \;=\; \sum_{h \in H} \delta_l(\tilde{x}, h\tilde{y})$$

where $\tilde{x}$ is some lift of x and $\tilde{y}$ is some lift of y.

Then we have

Lemma 4 *For each integer $l \geq 0$, the trace of D_l is determined by the spectrum of Γ. The numbers $\operatorname{Tr} D_1, \operatorname{Tr} D_2, \ldots, \operatorname{Tr} D_N$ determine the spectrum of Γ.*

Proof Since D_l comes from a point-pair invariant, by Corollary 3, its trace is determined by the spectrum of Γ.

For the second assertion, by Lemma 2, we have

$$\operatorname{Tr} D_l = \sum_{i=1}^{N} S_{\lambda_i}(l). \tag{6}$$

Because $S_{\lambda_i}(l)$ is a fixed l^{th}-degree polynomial in λ_i, the right side of (6) is a linear expression in the quantities $\sum \lambda_i, \sum \lambda_i^2, \ldots, \sum \lambda_i^l$, all sums taken as i goes from 1 to N. For each power q, let m_q denote the sum $\sum \lambda_i^q$, which is known as the q^{th} *moment* of the eigenvalues.

If we know $\operatorname{Tr} D_1, \operatorname{Tr} D_2, \ldots, \operatorname{Tr} D_N$, then for each $l = 1, 2, \ldots, N$, equation (6) may be viewed as a linear equation in some subset of the unknowns $m_1, m_2, \ldots, m_l$. We have N such equations, and they are all independent, because for each l, the coefficient of m_l in the equation for $\operatorname{Tr} D_l$ is non-zero, and the coefficients of all higher moments are zero. Thus we can solve the system for the moments $m_1, m_2, \ldots, m_N$. Then by Newton's formulas (see [8, p. 260]), we can reconstruct the spectrum from its first N moments. □

The trace of D_l has geometric significance, as well. Let x_0 be a vertex in $\Gamma = H \backslash X_k$ and $\tilde{x}_0$ be some lift of x_0 in the covering graph X_k. If $\tilde{x}_0'$ is some other lift of x_0 at a distance l from $\tilde{x}_0$, then the unique NBT l-walk from $\tilde{x}_0$ to $\tilde{x}_0'$ descends to a closed NBT l-walk at x_0 in Γ. Conversely, each closed NBT l-walk at x_0 in Γ has a lift which is an NBT walk in X_k beginning at $\tilde{x}_0$ and ending at some vertex $\tilde{x}_0'$ at a distance l from $\tilde{x}_0$.

Thus the NBT closed l-walks at x_0 in Γ are in one-to-one correspondence with the lifts of x_0 at a distance l from a particular lift $\tilde{x}_0$ in X_k. The number of such lifts is just $D_l(x_0, x_0)$. It follows that the trace of D_l counts the total number of NBT closed l-walks in Γ. In summary, we have

Theorem 5 *For each l, let d_l denote the total number of* NBT *closed l-walks in a N-vertex, k-regular graph Γ. Then for each l, $d_l = \operatorname{Tr} D_l$; the num-*

bers d_l are determined by the spectrum of Γ; and the numbers $d_1, d_2, \ldots, d_N$ determine the spectrum of Γ.

6 Tree automorphisms and cycles

Let G denote the full automorphism group of the tree X_k. The following classification of elements of G is found in [3].

Lemma 6 *Let $g \in G$ be different from the identity. Then exactly one of the following three cases holds.*

1. *The element g fixes some vertex. In this case, g is called a* rotation*.*
2. *The element g fixes some edge e, interchanging the two vertices on which e is incident. In this case, g is called an* inversion*.*
3. *There is some geodesic γ in X_k such that $g(\gamma) = \gamma$, and some positive integer n such that for each $x \in \gamma$, $\operatorname{dist}(x, g(x)) = n$. In this case, g is called a* translation*.*

More precisely, in case 3, g is called a *length-n translation along* γ. The geodesic γ is called the *axis* of g. It is easy to check that if g is a length-n translation, then $\operatorname{dist}(x, g(x)) \geq n$ for all $x \in V(X_k)$ with equality if and only if x is on the axis of g. We will also make use of the fact that if g_1 and g_2 are translations along the geodesic γ with respective lengths n_1 and n_2, then $g_1 g_2$ is a translation along γ of length either $n_1 + n_2$ or $|n_1 - n_2|$. (In the latter case, if $n_1 = n_2$, then $g_1 g_2$ is a rotation, rather than a translation.)

Let H be a subgroup of G such that $H \backslash X_k$ is a finite graph with no loops or parallel edges. This implies that each non-trivial element of H is a translation of length at least 3. In fact, H can be identified with the free homotopy group of $H \backslash X_k$. We now explore this identification in some detail.

Let $h \in H$ be a length-n translation, and pick a vertex x_0 on the axis of h. The unique NBT n-walk

$$(x_0, e_1, x_1, e_2, x_2, \ldots, x_{n-1}, e_{n-1}, x_n)$$

from x_0 to $x_n = h(x_0)$ in X_k descends to a closed NBT n-walk W in Γ. Moreover, W must be tailless, for if x_1 and x_{n-1} descend to the same vertex, then H contains an element h' with $h'(x_{n-1}) = x_1$. Then $h'h(x_0)$ is a neighbor of x_1. If $h'h(x_0) = x_0$, then we must have $h' = h^{-1}$. But then h' would be a length-n translation, which is impossible, since

$$\text{dist}(x_{n-1}, h'(x_{n-1})) = \text{dist}(x_{n-1}, x_1) = n - 2.$$

So we must have $\text{dist}(x_0, h'h(x_0)) = 2$, which implies $h'h$ is a translation of length at most 2. But this too is impossible, since H contains no translations of length less than 3. Thus the images of x_1 and x_{n-1} must be distinct, and the walk W is a tailless closed NBT n-walk.

That is, the image of the n-walk

$$(x_0, e_1, x_1, e_2, x_2, \ldots, x_{n-1}, e_{n-1}, x_n)$$

along the axis of h represents an n-cycle in $H\backslash X_k$. In fact, since the axis of h is made up of translations of this n-walk by powers of h, it is clear that any NBT n-walk along the axis of h descends to a representative of the same n-cycle, and we have a well-defined map from elements of H to cycles in $H\backslash X_k$.

Let $[H]$ denote the set of conjugacy classes in H. For $h \in H$, let $[h]_H$ denote the H-conjugacy class of h in H. We define a map Φ from $[H]$ to the set of cycles in $H\backslash X_k$ as follows. For $h \in H$, let γ_h be the axis of h in X_k, and map the conjugacy class $[h]_H$ to the cycle represented by the image of γ_h in $H\backslash X_k$.

Lemma 7 *The map Φ is well-defined, one-to-one, and onto.*

Proof To see that Φ is well-defined, we observe that if $h \in H$ is a translation of length n and $\alpha \in H$, then $\alpha h \alpha^{-1}$ is also a translation of length n. Furthermore, if γ_h is the axis of h, then the axis of $\alpha h \alpha^{-1}$ is $\alpha(\gamma_h)$. Let $(x_0, e_1, x_1, e_2, \ldots, x_{n-1,} e_n, x_n)$ be an NBT n-walk along γ_h and let W be its image in $H\backslash X_k$. Then

$$(\alpha(x_0), \alpha(e_1), \alpha(x_1), \ldots, \alpha(x_{n-1}), \alpha(e_n), \alpha(x_n))$$

descends to the same walk W, since each $\alpha(x_i)$ or $\alpha(e_i)$ differs from the corresponding x_i or e_i by an element α of the covering group.

To see that Φ is one-to-one, suppose that h_1 and h_2 are elements of H, and their respective axes γ_{h_1} and γ_{h_2} descend to representatives of the same n-cycle in $H\backslash X_k$. Let x and y be two adjacent vertices on γ_{h_1}. Then there is some $\alpha \in H$ such that $\alpha(x)$ is on γ_{h_2}. Because γ_{h_1} and γ_{h_2} cover the same n-cycle, one of the neighbors of $\alpha(x)$ must be an H-translate of y, so there is some $\beta \in H$ such that $\beta(y)$ is on γ_{h_2} and adjacent to $\alpha(x)$. Then $\beta^{-1}\alpha(x)$ must be adjacent to y. If $\beta^{-1}\alpha(x) \neq x$, then $\text{dist}(x, \beta^{-1}\alpha(x)) = 2$ which cannot occur, so we must have $\beta^{-1}\alpha(x) = x$. Since H contains no rotations, we must have $\beta = \alpha$.

Repeating this argument for each pair of adjacent vertices along γ_{h_1} shows that there is a single element $\alpha \in H$ taking γ_{h_1} to γ_{h_2}. Since h_1 and h_2 have the same length, $\alpha h_2 \alpha^{-1}$ and h_1 must agree on some non-empty set of vertices in γ_{h_1}, and since H contains no rotations, we may conclude that $\alpha h_2 \alpha^{-1} = h_1$.

To see that Φ is onto, let $\langle W \rangle$ be an n-cycle in $H\backslash X_k$ represented by a walk

$$W = (x_0, e_1, x_1, \ldots, x_{n-1}, e_n, x_n = x_0).$$

Let $\tilde{W}$ be a lift of W to X_k and let $\tilde{x}_0$ and $\tilde{x}_n$ denote the first and last vertices in $\tilde{W}$. Then there is a unique element $h \in H$ such that $h(\tilde{x}_0) = \tilde{x}_n$. We claim that the axis of h contains $\tilde{x}_0$ and $\tilde{x}_n$, and thus $\tilde{W}$. If $\tilde{x}_0$ were some positive distance r off the axis of h, then $\tilde{x}_n$ would be the same distance off the axis of h, and the projection of $\tilde{W}$ would have a tail. Since $\tilde{W}$ is tailless, it must be that $\tilde{W}$ is on the axis of h, and thus that $\Phi([h]_h) = \langle W \rangle$. $\square$

7 Centralizers and primitive cycles

For $h \in H$, let $C_H(h)$ denote the centralizer of h in H.

Lemma 8 *$C_H(h)$ is cyclic.*

Proof Suppose h is a length-n translation along a geodesic γ. Let $g \in C_H(h)$ and let $x \in \gamma$. Then $\text{dist}(x, h(x)) = n$, and since g is an automorphism,

$\text{dist}(g(x), gh(x)) = n$. Now $gh = hg$, so we have $\text{dist}(g(x), hg(x)) = n$, which shows that $g(x) \in \gamma$. It follows that $g(\gamma) \subset \gamma$, and thus that g is a translation along γ.

Since $C_H(h)$ contains no non-trivial rotations and every element of $C_H(h)$ fixes γ, $C_H(h)$ can contain at most one translation of any given length. The element g of $C_H(h)$ with the smallest length must then generate all of $C_H(h)$. $\square$

An n-cycle $\langle W \rangle$ is called *primitive* if a representative W of $\langle W \rangle$ does not make $m > 1$ trips around a representative of some n/m-cycle. If an n-cycle represented by

$$W = (x_0, e_1, x_1, \ldots, x_{n-1}, e_n, x_n)$$

is not primitive, then there is some least index j (a factor of n) such that $x_i = x_{j+i}$ and $e_i = e_{j+i}$ for all i. The walk

$$W' = (x_0, e_1, x_1, \ldots, x_{j-1}, e_j, x_j)$$

represents a primitive j-cycle, called the *primitive part* of the cycle $\langle W \rangle$.

Lemma 9 *Let $h \in H$ and let $\langle W \rangle = \Phi([h]_H)$ be the cycle in $H \backslash X_k$ corresponding to h. If g is a generator of $C_H(h)$, then $\Phi([g]_H)$ is the primitive part of $\langle W \rangle$.*

Proof Let $\langle W' \rangle$ be the primitive part of $\langle W \rangle$ and suppose $\langle W' \rangle$ is a j-cycle. Then there is a $g \in \Phi^{-1}(\langle W' \rangle)$ that is a length-j translation along the axis of h. Since $ghg^{-1}h^{-1}$ fixes the axis of h (and is not a rotation), $ghg^{-1}h^{-1}$ must be the identity, so $g \in C_H(h)$.

We claim that $C_H(h)$ is generated by g. If not, then $C_H(h)$ contains some translation g' along the axis of h (which is also the axis of g) having length k dividing j, and such that $(g')^{j/k} = g$. But that would imply that $\Phi([g]_H)$ consists of j/k trips around $\Phi([g']_H)$, contradicting the assumption that $\Phi([g]_H)$ is primitive. Thus $C_H(h)$ is indeed generated by g. $\square$

For $h \in H$, we denote by $L(C_H(h))$ the length of a generator of the centralizer $C_H(h)$ of h, or, equivalently, the length of the primitive part of the cycle $\Phi([h]_H)$.

As a corollary to the proof of Lemma 9, we remark that if h' and h are conjugate in H, then $L(C_H(h)) = L(C_H(h'))$. This also follows easily from the fact that $C_H(h)$ and $C_H(h')$ are themselves conjugate subgroups of H, so that $L(C_H(h))$ depends only on the conjugacy class of h.

8 An isospectrality condition

Lemma 10 *Let $\langle W \rangle$ be an n-cycle in a k-regular graph $H \backslash X_k$, and let j be the length of the primitive part of $\langle W \rangle$.*

1. *There are j distinct closed, tailless* NBT *n-walks in $H \backslash X_k$ that represent the cycle $\langle W \rangle$.*
2. *For each integer $r \geq 1$, there are $j(k-2)(k-1)^{r-1}$ distinct closed* NBT *$(n+2r)$-walks in $H \backslash X_k$ whose tailless parts represent the cycle $\langle W \rangle$.*

Proof

1. The tailless, closed NBT n-walks representing $\langle W \rangle$ correspond to the vertices in the cycle $\langle W \rangle$ where such a walk could begin. The number of such vertices yielding distinct walks is equal to the number of vertices in the primitive part of $\langle W \rangle$.
2. We get a closed NBT $(n+2r)$-walk whose tailless part represents $\langle W \rangle$ by attaching a tail of length r to some vertex in $\langle W \rangle$. The number of length-r tails which may be attached to a vertex of $\langle W \rangle$ without backtracking is $(k-2)(k-1)^{r-1}$. The number of sites on $\langle W \rangle$ where attaching length-r tails yields distinct $(n+2r)$-walks is again equal to the number of vertices in the primitive part of $\langle W \rangle$.

□

Let $n \geq 3$ be an integer, and suppose H contains a length-n translation t_n. Let $[t_n]_G$ denote the G-conjugacy class of t_n in G. By [4], Proposition 2.9, we know $[t_n]_G$ contains every length-n translation in H. The set $[t_n]_G \cap H$ is a union of finitely many H-conjugacy classes $[h_i]_H$ in H.

If $[t_n] \cap H$ is non-empty, let

$$P_n = \sum_{[h_i]_H \subset [t_n]_G} L(C_H(h_i)).$$

If $[t_n] \cap H$ is empty, set $P_n = 0$.

By Lemma 7 and Lemma 9, the number P_n may be viewed as the sum over all n-cycles $\langle W \rangle$ in $H \backslash X_k$ of the length of the primitive part of $\langle W \rangle$. That is, P_n is equal to the number of sites in $H \backslash X_k$ where we may attach a tail (possibly of length 0) to an n-cycle to get a closed NBT walk.

Since every closed NBT walk in $H \backslash X_k$ arises from some cycle, we can now use Lemma 10 to count all the closed NBT walks in $H \backslash X_k$. For each length n, $H \backslash X_k$ contains P_n closed NBT n-walks and $(k-2)(k-1)^{r-1} P_n$ closed NBT $(n+2r)$-walks for each $r \geq 1$.

Looking at this another way, we have the following result.

Lemma 11 *Letting d_n denote the total number of closed* NBT *n-walks in $H \backslash X_k$, and P_n be as above, we have*

$$d_n = P_n + (k-2)P_{n-2} + (k-1)(k-2)P_{n-4} + (k-1)^2(k-2)P_{n-6} + \cdots$$

where the sum terminates with the P_3 or P_4 term, according to the parity of n.

Thus for each $n \geq 1$ the numbers $P_1, P_2, P_3, \ldots, P_n$ (the first two of which are 0) determine the numbers $d_1, d_2, d_3, \ldots, d_n$. In fact, the equations in Lemma 11 giving the d_n in terms of the P_k for $k \leq n$ are all independent, and each is linear in the P_k, so the system may be solved for the P_k. That is, the numbers $d_1, d_2, d_3, \ldots, d_n$ determine the numbers $P_1, P_2, P_3, \ldots, P_n$.

Since the numbers d_n determine and are determined by the spectrum of the graph, we have established our main result.

Theorem 12 *Let $H\backslash X_k$ be an N-vertex, k-regular graph with no loops or parallel edges. For each integer $n \geq 1$, let*

$$P_n = \sum_{[h_i]_H \subset [t_n]_G} L(C_H(h_i))$$

where $[t_n]_G$ is the G-conjugacy class containing all length-n translations in H and $L(C_H(h_i))$ denotes the length of a generator of the centralizer $C_H(h_i)$ of h_i in H. Then the spectrum of $H\backslash X_k$ determines and is determined by the sequence $P_1, P_2, \ldots, P_N$.

9 Postscript: The Ihara zeta function

We remark that the numbers P_n that appear in our Theorem 12 also turn up in a somewhat different approach to the same subject.

In [6] we find a discussion of the Ihara zeta function $Z(u)$ associated with a graph Γ, given by

$$Z(u) = \prod (1 - u^{L(\langle W \rangle)})^{-1}$$

where the product is taken over all primitive cycles $\langle W \rangle$ in Γ, and $L(\langle W \rangle)$ denotes the length of the cycle $\langle W \rangle$.

If Γ is k-regular and $\{\lambda_1, \lambda_2, \ldots, \lambda_N\}$ is the spectrum of Γ, then it can be shown that the Ihara zeta function has the spectral expression

$$Z(u) = \left[(1-u^2)^{r-1} \prod_i (1 - \lambda_i + (k-1)u^2) \right]^{-1}$$

where r is the rank of the fundamental group of Γ.

The numbers P_n appear when we consider the logarithmic derivative of $Z(u)$. Specifically, we have

$$u \frac{d}{du} \log Z(u) = \sum_{\langle W \rangle \text{ primitive}} L(\langle W \rangle) \sum_{j \geq 1} u^{jL(\langle W \rangle)}.$$

The coefficient of u^n in this expression is a sum of the lengths of all the primitive cycles whose lengths divide n. That is, it is the number of distinct sites where a tail, possibly of length 0, could be attached to an n-cycle (primitive or not) in Γ. This quantity is exactly our number P_n, and so

$$u\frac{d}{du}\log Z(u)$$

may be viewed as a generating function for the P_n.

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