Combinatorics of free product graphs

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Abstract

We define the return generating function on an abstract graph, and develop tools for computing such functions. The relation between a graph's return generating function and its spectrum is discussed.

1 Introduction

In his 1986 dissertation "Random walks and convolution operators on free products" [4], John C. McLaughlin computes the spectrum of the Cayley graph of the free product group $\mathbb{Z}/(2) \star \mathbb{Z}/(3)$, putting to use a theorem he proves about the relations among the Green's functions on the Cayley graph of a free product and the Cayley graphs of the factor groups.

To state this theorem in McLaughlin's notation, let A and B be finitely-generated groups, and for any group C let

$$G^{(C)}(w) = [(wI - A)^{-1}\delta_e](e)$$

where A is the adjacency operator on a Cayley graph of C, I is the identity operator, and δ_e is the delta function at the identity element e. Assume we have fixed, undirected Cayley graphs for A and B. Then McLaughlin's theorem says

$$\begin{aligned}
G^{(A\star B)}(w) &= G^{(A)}(w - s_B(w)) \\
G^{(A\star B)}(w) &= G^{(B)}(w - s_A(w)) \\
G^{(A\star B)}(w) &= \frac{1}{1 - s_A(w) - s_B(w)}
\end{aligned}$$

where $s_A(w)$ and $s_B(w)$ are functions which can be determined by solving this system.

The Green's function $G^{(C)}(w)$ on a Cayley graph is closely related to the combinatorial return generating function on the same graph. Our purpose in this paper is to present a combinatorial form of McLaughlin's theorem, stated in terms of return generating functions. In doing so, we will generalize somewhat the class of graphs to which this technique applies. This generalization will be very slight, however. Our principal goal is to provide a purely combinatorial proof of a theorem equivalent to that stated above.

In section 2 we establish notation and discuss the adjacency operator on an abstract graph. In section 3 we define the connected sum of two graphs and develop techniques (theorem 3.2 and corollary 3.3) for computing the return generating function on a finite graph. In section 4 we examine the Cayley graph of a free product, and define the free product of general graphs. We then prove theorem 4.9, our combinatorial version of McLaughlin's theorem, and apply it to a few examples.

2 The setting

2.1 Walks

Let Γ be an undirected graph.

Definition 2.1 *A walk of length n in Γ is a sequence of vertices and edges*

$$v_1\epsilon_1 v_2\epsilon_2 \cdots v_n\epsilon_n v_{n+1}$$

in which each edge ϵ_i is incident at v_i and v_{i+1}. If $v_1 = v_{n+1}$, the walk is called a closed walk of length n at v_1. *Otherwise, it is called a walk of length n from v_1 to v_{n+1}.*

Note that a walk is completely determined by its initial vertex and its edge sequence. If Γ is not a multigraph, then a walk is determined, also, by its vertex sequence.

We make no restrictions on the vertices and edges of a walk beyond those in the definition. In particular, any vertex or edge may occur any number of times in a walk. Viewing a walk informally as the path followed by some graph-dwelling creature that steps from vertex to vertex along edges, we remark that a walk may "double back on itself" any number of times.

2.2 Rooted graphs

In most of our graphs, we will want to have a distinguished vertex to serve as a sort of "home base" for our graph-dwelling creature. We will call this distinguished vertex the *root* of the graph, and make the following definition.

Definition 2.2 *A rooted graph is a pair* (Γ, e)*, where* Γ *is a graph and* e *is a vertex in* Γ*.*

If Γ is a Cayley graph, for example, we will normally choose for e the vertex corresponding to the identity.

2.3 Return generating functions

Given a rooted graph (Γ, e), we are interested in counting, for each non-negative integer n, the number of closed walks of length n in Γ at e. We will do so by means of a generating function.

Definition 2.3 *The return generating function for* Γ *at* e *is the power series*

$$R_e(z) = \sum_{n=0}^{\infty} \rho_n z^n$$

where ρ_n *is the number of closed walks of length* n *in* Γ *at* e*.*

A closed walk in Γ at e may visit the vertex e many times. A close relative of the return generating function is the first-return generating function, which counts the number of walks which begin at e and return to e for the first time after exactly n steps.

Definition 2.4 *The first-return generating function for* Γ *at* e *is the power series*

$$Q_e(z) = \sum_{n=0}^{\infty} \zeta_n z^n$$

where ζ_n *is the number of walks of length* n *in* Γ *in which* e *occurs as the first vertex and the last vertex, but never as an intermediate vertex. For convenience, we set* $Q_e(0) = 0$*.*

Here and throughout this paper, we will regard generating functions as formal power series, without worrying about questions of convergence. Our generating functions will not be "functions" so much as simple combinatorial gadgets, useful because of the formal properties of power series multiplication and addition.

The power series $R_e(z)$ and $Q_e(z)$ are related by the following theorem.

Theorem 2.5 *Let* (Γ, e) *be a rooted graph with return generating function* $R_e(z)$ *and first-return generating function* $Q_e(z)$*. Then*

$$R_e(z) = \frac{1}{1 - Q_e(z)} \tag{1}$$

$$Q_e(z) = 1 - \frac{1}{R_e(z)}. \tag{2}$$

PROOF. Every closed walk of length n at e is made up of a unique sequence of first-return closed walks at e whose aggregate length is n. Thus we can count all closed walks of length n at e by summing sequences of first-return closed walks over all possible ordered partitions of the number n. Specfically, we have

$$\rho_n = \sum_{\gamma_1+\gamma_2+\cdots+\gamma_k=n} \zeta_{\gamma_1}\zeta_{\gamma_2}\cdots\zeta_{\gamma_k} \tag{3}$$

where the γ_i are positive integers (recall that $\zeta_0 = 0$). If we adopt the convention that the only partition of $n = 0$ is the empty partition, and that the empty product is equal to 1, then equation (3) works for all n. Observing that there are no partitions of n with length less than 1 (unless $n = 0$) or greater than n, we can group the terms in the sum above by the length of the partition, without having to be too careful about summation limits.

$$\rho_n = \sum_{p=0}^{\infty} \left(\sum_{\gamma_1+\gamma_2+\cdots+\gamma_p=n} \zeta_{\gamma_1}\zeta_{\gamma_2}\cdots\zeta_{\gamma_p} \right). \tag{4}$$

Raising $Q_e(z)$ to the power p, we have

$$\begin{aligned} (Q_e(z))^p &= \left(\sum_{n=0}^{\infty} \zeta_n z^n \right)^p \\ &= \sum_{n=0}^{\infty} \left(\sum_{\gamma_1+\gamma_2+\cdots+\gamma_p=n} \zeta_1\zeta_2\cdots\zeta_n \right) z^n. \end{aligned}$$

That is, the coefficient of z^n in the product $[Q_e(z)]^p$ is the sum of $\zeta_{\gamma_1}\zeta_{\gamma_2}\cdots\zeta_{\gamma_p}$ over all p-length partitions $\gamma_1+\gamma_2+\cdots+\gamma_p$ of n. Thus equation (4) tells us that ρ_n is the sum over all p of the coefficient of z^n in $[Q_e(z)]^p$. This makes ρ_n equal to the coefficient of z^n in

$$1+Q_e(z)+[Q_e(z)]^2+\cdots=\frac{1}{1-Q_e(z)}.$$

Two generating functions with the same coefficients are equal, so we have established claim (1) above. Equation (2) is simply equation (1), solved for $Q_e(z)$. □

2.4 Connected sums

Let (Γ_1,e_1) and (Γ_2,e_2) be rooted graphs. We form the connected sum $(\Gamma_1,e_1)\sharp(\Gamma_2,e_2)$ by gluing the vertices e_1 and e_2 together. The new graph looks like Γ_1 with a copy of Γ_2 hanging off it at e_1, or like Γ_2 with a copy of Γ_1 hanging off it at e_2, depending on one's point of view.

To be precise, we will make the following definition.

Definition 2.6 *Let (Γ_1,e_1) and (Γ_2,e_2) be rooted graphs. Let V_i and E_i be the vertex and edge sets of Γ_i. For $i=1,2$, let E_i' be the set formed from E_i by replacing every occurrence of e_i with a new symbol e. Let V_i' be the set V_i with the element e_i removed. Then the graph $(\Gamma_1,e_1)\sharp(\Gamma_2,e_2)$ has vertex set $V_1'\cup V_2'\cup\{e\}$ and edge set $E_1'\cup E_2'$. The root vertex of $(\Gamma_1,e_1)\sharp(\Gamma_2,e_2)$ is e.*

The connected sum operator is obviously commutative and associative, so an expression like

$$\sharp_{i=1}^{k}(\Gamma_i,e_i)$$

makes sense.

When no mistake can be made about which vertex of a rooted graph is the root, we may slip back into less careful notation, and write expressions like

$$\sharp_{i=1}^{k}\Gamma_i.$$

We have the following tools to help us count closed walks in such a connected sum.

Theorem 2.7 *Let* (Γ_i, e_i) *be rooted graphs for* $i = 1, \ldots, k$. *Let* $Q_{e_i}(z)$ *be the first-return generating function for* (Γ_i, e_i). *Then*

1. *The first-return generating function* $Q_e(z)$ *for the connected sum of all the* (Γ_i, e_i) *is*
$$Q_e(z) = \sum_{i=1}^{k} Q_{e_i}(z).$$
2. *The return generating function* $R_e(z)$ *for the connected sum of all the* (Γ_i, e_i) *is*
$$R_e(z) = \frac{1}{1 - \sum_{i=1}^{k} Q_{e_i}(z)}$$

Proof. A first-return closed walk in a connected sum must stay in only one of the summand graphs, because the summand graphs are interconnected only through the root vertex. Thus every first-return closed walk in a connected sum is in fact a first-return closed walk in one of the summand graphs. Each function $Q_{e_i}(z)$ counts the first-return paths in a separate summand graph, so by the addition principle, the sum of the $Q_{e_i}(z)$ counts the first-return paths in all the summand graphs. The second assertion follows from the first by applying theorem 2.5. □

2.5 Symmetric graphs

The computation of $R_e(z)$ will be our main activity in this paper. We know of one case where finding an explicit expression for $R_e(z)$ is quite straightforward.

Definition 2.8 *A graph* Γ *is symmetric if, for every pair of vertices* x *and* y, *there is an isomorphism of* Γ *which takes* x *to* y.

That is, a symmetric graph looks the same from any vertex. A Cayley graph, for example, is always symmetric. In the next section, we show how to find an explicit expression for $R_e(z)$ when Γ is symmetric and finite.

3 $R_e(z)$ on a finite graph

3.1 The adjacency operator and its spectrum

Let Γ be an undirected graph.

By a *function* on Γ, we mean a mapping from the vertices of Γ to the real numbers. Thus $L^2(\Gamma)$ is a vector space over $\mathbb{R}$, with basis $\{\delta_x : x \text{ is a vertex in } \Gamma\}$, where the delta function δ_x is given by

$$\delta_x(y) = \begin{cases} 1 & \text{if } x = v \\ 0 & \text{otherwise.} \end{cases}$$

If the number of vertices in Γ is a finite number N, then $L^2(\Gamma)$ is isomorphic to the usual Euclidean space $\mathbb{R}^N$. We define the inner product $\langle \cdot, \cdot \rangle$ on $L^2(\Gamma)$ in the correspondingly usual way: if f and g are functions on Γ, then

$$\langle f, g \rangle = \sum_{x \in \Gamma} f(x) g(x)$$

where the sum is taken over all the vertices in Γ.

For each pair of vertices x and y in Γ, let $C(x,y)$ denote the number of edges in Γ whose endpoints are x and y. (If Γ is not a multigraph, this number will always be zero or one.) Define the operator A on $L^2(\Gamma)$ by

$$(Af)(x) = \sum_{y \in \Gamma} C(y,x) f(y).$$

We claim that the operator A is self-adjoint. This follows immediately from the fact that $C(x,y) = C(y,x)$ for all x and y, which is nothing more than the statement that Γ is undirected. To carry this out in some detail, let $f, g \in L^2(\Gamma)$. Then

$$\begin{aligned} \langle Af, g \rangle &= \sum_{x \in \Gamma} \sum_{y \in \Gamma} C(y,x) f(y) g(x) \\ &= \sum_{y \in \Gamma} f(y) \sum_{x \in \Gamma} C(y,x) g(x) \\ &= \sum_{y \in \Gamma} f(y) \sum_{x \in \Gamma} C(x,y) g(x) \\ &= \langle f, Ag \rangle. \end{aligned}$$

Now let us assume that Γ is finite, with N vertices, so that A is a linear operator on the space $\mathbb{R}^N$. Because A is self-adjoint, it is unitarily equivalent to a real diagonal operator. That is, there exist functions $\varphi_1, \varphi_2, \ldots, \varphi_N$ and real numbers $\lambda_1, \lambda_2, \ldots, \lambda_N$ such that the φ_i are an orthonormal basis for $L^2(\Gamma)$, and for each i,

$$A\varphi_i = \lambda_i \varphi_i.$$

The set of eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_N\}$ is usually called the *spectrum of* Γ, without explicit reference to the operator A.

If we identify $L^2(\Gamma)$ with the Euclidean space $\mathbb{R}^N$ by numbering the vertices $x_1, x_2, \ldots, x_N$ in some arbitrary manner, and then mapping each delta function δ_{x_i} to a basis vector $\mathbf{e}_i$ of $\mathbb{R}^N$, we can write the operator A in matrix form. The $(i,j)^{\text{th}}$ entry of A is equal to $C(x_i, x_j)$, the number of edges in Γ whose endpoints are x_i and x_j. That is, the operator A in matrix form is simply an adjacency matrix of Γ, and to find the spectrum of Γ, we need only write down this matrix and diagonalize it.

3.2 Spectral formulas for $R_e(z)$

We will write the return generating function $R_e(z)$ for Γ in terms of the λ_i and φ_i, but first, we need to state the relation between the operator A and walks in Γ.

Lemma 3.1 *Let x and y be vertices in an undirected graph Γ. Let $\rho_n(x,y)$ denote the number of walks of length n in Γ from x to y. Then*

$$\rho_n(x,y) = \langle \delta_x, A^n \delta_y \rangle.$$

Proof. We proceed by induction. In the base case, when $n = 0$, the left-hand side is (by definition) 1 if $x = y$ and 0 otherwise. The right-hand side reduces to $\langle \delta_x, \delta_y \rangle$, which is also 1 if $x = y$ and 0 otherwise.

Now assume $n \geq 1$ and $\rho_{n-1}(x,y) = \langle \delta_x, A^{n-1}\delta_y \rangle$ for each pair of vertices x and y. Consider a walk of length n from x to y. Its first step goes from x along some edge to a neighbor z of x. The remainder of the walk is a walk of length $n-1$ from z to y. Thus, one way to construct any walk of length n from x to y is to choose an edge by which to leave x, and then (independently) choose a walk of length $n-1$ from the other end of that edge to the vertex y. This combinatorial argument gives us the following expression for ρ_n:

$$\begin{aligned}\rho_n(x,y) &= \sum_{z\in\Gamma} C(x,z)\rho_{n-1}(z,y) \\ &= \sum_{z\in\Gamma} C(x,z)\langle\delta_z, A^{n-1}\delta_y\rangle \\ &= \sum_{z\in\Gamma} C(x,z)(A^{n-1}\delta_y)(z) \\ &= [A(A^{n-1}\delta_y)](x) \\ &= \langle\delta_x, A^n\delta_y\rangle\end{aligned}$$

as required. □

We are now ready to compute $R_e(z)$ for a finite graph.

Theorem 3.2 *Let Γ be a finite graph. Let $\{\lambda_1, \lambda_2, \ldots, \lambda_N\}$ be its spectrum, and let $\{\varphi_1, \varphi_2, \ldots, \varphi_N\}$ be an orthonormal set of eigenfunctions for A, such that $A\varphi_i = \lambda_i\varphi_i$. Let e be a vertex in Γ. Then*

$$R_e(z) = \sum_{i=1}^{N} \frac{\varphi_i^2(e)}{1-\lambda_i z}.$$

Proof. By lemma 3.1, we know that

$$\rho_n = \rho_n(e,e) = \langle\delta_e, A^n\delta_e\rangle \tag{5}$$

Because the eigenfunctions φ_i span $L^2(\Gamma)$, we can write δ_e as a linear combination of the φ_i, as follows:

$$\begin{aligned}\delta_e &= \sum_{i=1}^{N} \langle\delta_e, \varphi_i\rangle\varphi_i &\quad (6)\\ &= \sum_{i=1}^{N} \varphi_i(e)\varphi_i. &\quad (7)\end{aligned}$$

We now substitute expression (7) for the second δ_e in equation (5), to get

$$\rho_n = \langle\delta_e, A^n \sum_{i=1}^{N} \varphi_i(e)\varphi_i\rangle$$

$$
\begin{aligned}
&= \langle \delta_e, \sum_{i=1}^{N} \lambda^n \varphi_i(e)\varphi_i \rangle \\
&= \sum_{i=1}^{N} \lambda_i^n \varphi_i^2(e).
\end{aligned}
$$

Next we substitute this expression for ρ_n into the definition of $R_e(z)$:

$$
\begin{aligned}
R_e(z) &= \sum_{n=0}^{\infty} \rho_n z_n \\
&= \sum_{n=0}^{\infty} \sum_{i=1}^{N} \lambda_i^n \varphi_i^2(e) z^n \\
&= \sum_{i=1}^{N} \left(\sum_{n=0}^{\infty} \lambda_i^n z^n \right) \varphi_i^2(e) \\
&= \sum_{i=1}^{N} \frac{\varphi_i^2(e)}{1 - \lambda_i z}.
\end{aligned}
$$

□

The appearance of the numbers $\varphi_i(e)$ in the formula in theorem 3.2 is somewhat irritating (since the φ_i are not as easy to find as the λ_i), but seems unavoidable; because we did not assume Γ to be symmetric, any formula for $R_e(z)$ must have some dependence on e. If, however, we include the assumption that Γ is symmetric, we can easily make this dependence go away.

Corollary 3.3 *Let Γ be a symmetric graph with a finite number N of vertices. Let $\{\lambda_1, \lambda_2, \ldots, \lambda_N\}$ be its spectrum. Let e be a vertex in Γ. Then*

$$
R_e(z) = \frac{1}{N} \sum_{i=0}^{N} \frac{1}{1 - \lambda_i z}.
$$

Proof. Let $\{\varphi_1, \varphi_2, \ldots, \varphi_N\}$ be an orthonormal set of eigenfunctions, corresponding to $\{\lambda_1, \lambda_2, \ldots, \lambda_N\}$. By theorem 3.2, for each vertex x in Γ, we have

$$
R_x(z) = \sum_{i=1}^{N} \frac{\varphi_i^2(x)}{1 - \lambda_i z}.
$$

We sum this expression over all vertices x to get

$$\begin{aligned} \sum_{x\in\Gamma} R_x(z) &= \sum_{x\in\Gamma}\sum_{i=1}^{N} \frac{\varphi_i^2(x)}{1-\lambda_i z} && (8)\\ &= \sum_{i=1}^{N} \frac{1}{1-\lambda_i z} \sum_{x\in\Gamma} \varphi_i^2(x) && (9)\\ &= \sum_{i=1}^{N} \frac{1}{1-\lambda_i z} && (10) \end{aligned}$$

because each φ_i is assumed to have norm 1.

Now, because Γ is symmetric, it looks the same from every vertex, and so the return function at any vertex is equal to the return function at e. This implies that

$$\sum_{x\in\Gamma} R_x(z) = N R_e(z),$$

so that

$$R_e(z) = \frac{1}{N} \sum_{x\in\Gamma} R_x(z).$$

Putting this together with equation 10 yields the result we have claimed.
□

3.3 Examples I

We will want some building blocks to play with later, so let's define some graphs and calculate their return generating functions.

Let F_k be the graph with $k+1$ vertices $v_0, v_1, \ldots, v_k$, and with edge set

$$\{(v_0, v_1), (v_0, v_2), \ldots, (v_0, v_k)\}.$$

(We think of F as in "fork".) Figure 1 shows pictures of F_3 and F_5. We will make F_k a rooted graph by declaring v_0 to be the root.

It is not difficult to calculate $R_{v_0}(z)$ for F_k using theorem 3.2. There is a $k-1$-dimensional eigenspace with eigenvalue 0 on F_k, spanned by eigenfunctions φ with $\varphi(v_0) = 0$ and $\sum_{i=1}^{k} \varphi(v_i) = 0$. The remaining two eigenvalues are

Figure 1: Fork graphs

$\pm\sqrt{k}$, and it is not hard to find orthonormal eigenfunctions corresponding to these eigenvalues. They are

$$\varphi_{\sqrt{k}}(v_0) = \frac{1}{\sqrt{2}}; \qquad \varphi_{\sqrt{k}}(v_i) = \frac{1}{\sqrt{2k}} \quad \text{for } i \neq 0$$
$$\varphi_{-\sqrt{k}}(v_0) = \frac{1}{\sqrt{2}}; \qquad \varphi_{-\sqrt{k}}(v_i) = -\frac{1}{\sqrt{2k}} \quad \text{for } i \neq 0.$$

The formula in theorem 3.2 gives us

$$\begin{aligned} R_{v_0}(z) &= \frac{\frac{1}{2}}{1-\sqrt{k}z} + \frac{\frac{1}{2}}{1+\sqrt{k}z} \\ &= \frac{1}{1-kz^2}. \end{aligned}$$

Of course, this return generating function could easily have been found by inspection, or even more easily, by observing that the first return generating function at v_0 is simply kz^2, and applying theorem 2.5.

For $k \geq 3$, let C_k denote the graph consisting of a single cycle of k vertices. C_3 and C_4 are shown in figure 2.

These are symmetric graphs, so we can find the return functions $R_e(z)$ at any vertex e using just their spectra.

The spectrum of C_3 is easily found to be $2, -1, -1$, and the spectrum of C_4 is $2, 0, 0, -2$. Thus, applying corollary 3.3, we have for C_3

$$\begin{aligned} R_e(z) &= \frac{1}{3}\left(\frac{1}{1-2z} + \frac{2}{1+z}\right) \\ &= \frac{z-1}{2z^2+z-1}, \end{aligned}$$

C_3 C_4

Figure 2: Cycle graphs

and for C_4

$$\begin{aligned} R_e(z) &= \frac{1}{4}\left(\frac{1}{1-2z} + 2 + \frac{1}{1+2z}\right) \\ &= \frac{2z^2-1}{4z^2-1}. \end{aligned}$$

The connected sum of F_3 and C_4 is shown in figure 3.

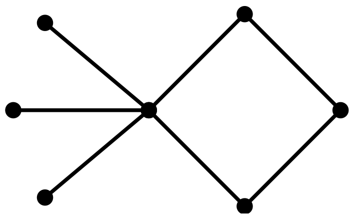


Figure 3: $F_3 \sharp C_4$

The root vertex e is the vertex shared by the two summand graphs. We can find $R_e(z)$ for this graph using theorem 2.7. We first note that the first return function $Q_{v_0}(z)$ for F_3 is $3z^2$. We can find the first return function $Q_e(z)$ for C_4 by using the formula (from theorem 2.5)

$$Q_e(z) = 1 - \frac{1}{R_e(z)}.$$

This gives us $Q_e(z) = \dfrac{2z^2}{1-2z^2}$ for C_4. Theorem 2.7 now gives us the return generating function for the connected sum as

$$R_e(z) = \frac{1}{1 - Q_{v_0}(z) - Q_e(z)} \tag{11}$$

$$= \frac{1-2z^2}{6z^4-7z^2+1}. \tag{12}$$

Since this function is equal, by theorem 2.5, to

$$\sum_{i=1}^{7} \frac{\varphi_i^2(e)}{1-\lambda_i z}$$

where $\{\lambda_1, \lambda_2, \ldots, \lambda_7\}$ is the spectrum of $F_3 \sharp C_4$, we know that the set

$$\{\lambda_1, \lambda_2, \ldots, \lambda_7\}$$

must contain the reciprocals of the roots of $6z^4 - 7z^2 + 1$. That is, the numbers ± 1 and $\pm\sqrt{6}$ are in the spectrum of $F_3 \sharp C_4$.

4 $R_e(z)$ on a free product

4.1 The Cayley graph of a free product

Let G_1 and G_2 be finitely-generated groups. Pick a finite, symmetric set of generators

$$S_1 = \{g_1, g_1^{-1}, g_2, g_2^{-1}, \ldots, g_k, g_k^{-1}\}$$

for G_1, and a set of relations R_1 among the g_i sufficient to form the presentation

$$G_1 = \langle S_1 : R_1 \rangle.$$

Let Γ_1 be the Cayley graph of G_1 with respect to the generators S_1. The symmetry of S_1 lets us view Γ_1 as an undirected graph.

Similarly, let S_2 be a finite symmetric set of generators for G_2 and form the presentation

$$G_2 = \langle S_2 : R_2 \rangle$$

and the Cayley graph Γ_2 of G_2 with respect to S_2.

The free product of G_1 and G_2, denoted $G_1 \star G_2$, is the group G with presentation

$$G = \langle S_1 \cup S_2 : R_1 \cup R_2 \rangle.$$

Let Γ be the Cayley graph of G with respect to the generators $S_1 \cup S_2$. What does Γ look like? From the identity vertex in Γ, edges lead out into two subgraphs, one of them a copy of Γ_1 and one a copy of Γ_2. At every other (i.e. non-identity) vertex x of this first copy of Γ_1, some edges lead out into an independent copy of Γ_2, glued to x by its identity vertex. Similarly, every vertex of the first copy of Γ_2 has a copy of Γ_1 glued to it by the identity vertex in Γ_1.

Every non-identity vertex of each of these new copies of Γ_2 has the identity vertex of an independent copy of Γ_1 glued to it, and every non-identity vertex of each new copy of Γ_1 has the identity vertex of a copy of Γ_2 glued to it. Continuing this gluing process indefinitely produces the Cayley graph of G.

To be precise, when we say a copy of Γ_1 is glued to some vertex x by its identity vertex, we mean that the identity vertex of Γ_1 is identified with the vertex x in the same way that the two vertices e_1 and e_2 are identified in the connected sum $(\Gamma_1, e_1)\sharp(\Gamma_2, e_2)$.

We will be able to form a somewhat more formal and more useful description of the Cayley graph Γ of $G_1 \star G_2$ by noting its resemblance to a tree, and defining it in terms of its branches.

First, we make the following assertion about the elements of the free product group $G_1 \star G_2$ (see [3]): Every non-identity element g of $G_1 \star G_2$ can be written in a unique way as a product

$$g = g_1 g_2 \cdots g_k$$

where each g_i is either a word in the generators of G_1 or a word in the generators of G_2, and for each i, g_i and g_{i+1} do not come from the same group. This is really just the statement that the product of G_1 and G_2 is free; there are no relations involving elements from both factor groups, so elements from G_1 can't ever cancel elements from G_2, and vice versa.

This allows us to define a mapping

$$F : G \longrightarrow \{G_1, G_2\}$$

which associates to each $g \in G$ the group which contributed its first non-trivial factor. Specfically, if $g = g_1 g_2 \cdots g_k$ then

$$F(g) = \begin{cases} G_1 & \text{if} \quad g_1 \in G_1 \\ G_2 & \text{if} \quad g_1 \in G_2. \end{cases}$$

We leave F undefined at the identity of $G_1 \star G_2$.

We will now make a provisional definition for the branches of the Cayley graph Γ. We will define them as subgraphs, and describe them in terms of their vertex sets. It will be understood that all possible edges are included in these subgraphs. That is, whenever x and y are vertices in a subgraph and there is an edge connecting x to y in the full graph Γ, that edge will be included in the subgraph as well.

Let B_1 be the subgraph of Γ whose vertex set comprises the identity vertex, together with all vertices x whose corresponding elements $g_x \in G$ satisfy

$$F(g_x) = G_1.$$

That is, B_1 is the Cayley graph of the subset $E_1 \subset G$ made up of all the elements of $G_1 \star G_2$ which "start out" in G_1, together with the identity. (We are using the term "Cayley graph" rather loosely here, but after all, this definition is only temporary.) What does B_1 look like? The identity vertex sits in a copy of Γ_1, because all the elements of G_1 are in E_1. The remaining elements of E_1 are of the form αg, where α is some nontrivial element of G_1 and g is any element of G whose first non-trivial factor is in G_2.

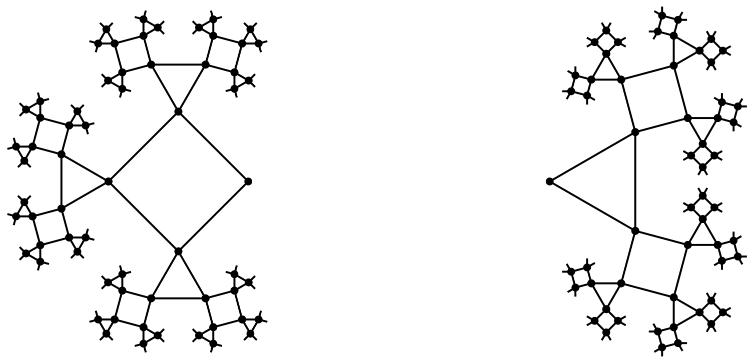
This means that for each non-identity element α of G_1, the vertex corresponding to α in our copy of Γ_1 has glued to it a subgraph which is isomorophic to the Cayley graph of the subset $E_2 \subset G$ given by

$$E_2 = \{g \in G : F(g) = G_2\} \cup \{\text{identity}\}.$$

Let us to denote this subgraph B_2. The answer to our question "What does B_1 look like?" is the following: B_1 consists of a copy of Γ_1 with a branch B_2 glued to each non-identity vertex.

The roles played by G_1 and G_2 in this discussion were entirely symmetric, so by interchanging 1's and 2's, we can answer the question that obviously follows: What does B_2 look like? It is a copy of Γ_2 with a branch B_1 glued to each non-identity vertex.

Figure 4 shows parts of the two branches of the free product graph which we will eventually call $C_3 \star C_4$. Since C_3 is a Cayley graph for $\mathbb{Z}/(3)$ and C_4 is a Cayley graph for $\mathbb{Z}/(4)$, we are thinking of this graph (for now) as a Cayley graph of the group $\mathbb{Z}/(3) \star \mathbb{Z}/(4)$. Branch 1 is a copy of C_4 with copies of branch 2 attached at all the non-identity vertices. Branch 2 is a copy of C_3 with copies of branch 1 attached at all the non-identity vertices. The full Cayley graph, shown in figure 5, is the connected sum of the two branches.



Branch 1 Branch 2

Figure 4: Branches of the graph $C_3 \star C_4$

4.2 The free product of Cayley graphs

This leads us to the following pair of recursive definitions for the branches B_1 and B_2. These definitions may seem a little suspicious, since they define B_1 and B_2 in terms of each other, but they do, in fact, completely describe the subgraphs, and will suit our purposes nicely in the next section.

Let Γ_1 and Γ_2 be Cayley graphs of G_1 and G_2 respectively.

Definition 4.1 *B_1 is the graph comprising a single copy of Γ_1 with an independent copy of B_2 glued, by its identity vertex, to every non-identity vertex in the copy of Γ_1.*

Definition 4.2 *B_2 is the graph comprising a single copy of Γ_2 with an independent copy of B_1 glued, by its identity vertex, to every non-identity vertex in the copy of Γ_2.*

We will now *define* the free product of the two Cayley graphs Γ_1 and Γ_2 to be the result of gluing B_1 and B_2 together.

Definition 4.3 *The free product $\Gamma_1 \star \Gamma_2$ is the graph*

$$B_1 \sharp B_2$$

where the branches B_1 and B_2 are defined as above, and the root vertices are taken to be the identity vertices.

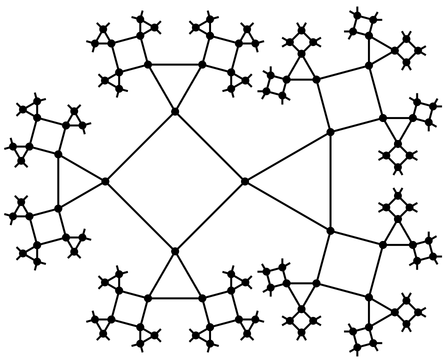


Figure 5: Cayley graph of $\mathbb{Z}/(3) \star \mathbb{Z}/(4)$

Looking back at the definition of B_1, we find another way to describe the free product $\Gamma_1 \star \Gamma_2$. We state it as an alternative to definition 4.3, claiming, of course, that the two definitions are equivalent.

Definition 4.4 *The free product* $\Gamma_1 \star \Gamma_2$ *is the graph formed by gluing one copy of* B_2*, by its identity vertex, to each vertex of the graph* Γ_1*.*

If we start with B_2, we get yet another equivalent definition.

Definition 4.5 *The free product* $\Gamma_1 \star \Gamma_2$ *is the graph formed by gluing one copy of* B_1*, by its identity vertex, to each vertex of the graph* Γ_2*.*

Figure 6 shows part of the Cayley graph of $\mathbb{Z}/(3) \star \mathbb{Z}/(4)$ as a copy of C_4 with copies of branch 2 glued to each vertex.

We have yet to show that the free product graph $\Gamma_1 \star \Gamma_2$, as we have defined it, is actually the Cayley graph Γ of the free product group $G_1 \star G_2$.

Theorem 4.6 *Let* $G = G_1 \star G_2$ *and let* Γ *be the Cayley graph of* G *with respect to the generators* $S_1 \cup S_2$*. Let* Γ_1 *and* Γ_2 *be the Cayley graphs of* G_1 *and* G_2 *with respect to their respective generating sets. Then*

$$\Gamma = \Gamma_1 \star \Gamma_2.$$

PROOF. In the preceding discussion, we established that B_1 is the subgraph of Γ containing all the vertices corresponding to elements of the subset $E_1 \subset$

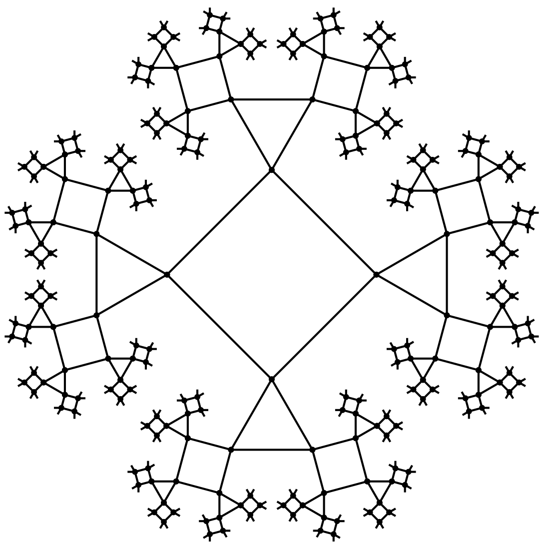


Figure 6: Part of $C_3 \star C_4$ with C_4 at the center

G, and B_2 is the subgraph of Γ containing all the vertices corresponding to elements of the subset $E_2 \subset G$. Since every element of G is in either E_1 or E_2, and only the identity is in both, the graph formed by gluing B_1 to B_2 at the identity vertex contains every vertex in Γ exactly once. It remains only to see that $\Gamma_1 \star \Gamma_2$ contains all the edges in Γ. By our earlier understanding about which edges are included in a subgraph, the only edges that could be missing from $\Gamma_1 \star \Gamma_2$ are those which connect a non-identity vertex x in B_1 to a non-identity vertex y in B_2. But Γ contains no such edges, because the group elements such as x and y always differ by an element of G which contains factors from both G_1 and G_2. Such an element cannot be in the symmetric generating set $S_1 \cup S_2$. $\square$

4.3 The free product of general graphs

The previous theorem notwithstanding, the fact that our original graphs Γ_1 and Γ_2 were Cayley graphs is becoming less and less important. In fact, in defining the free product of two graphs, the only group-theoretic notion that we used was that of the identity vertex. We can therefore define a free product of two general graphs, provided we come up with a substitute for the identity vertex. Such a substitute is built in to our notion of a rooted graph.

Furthermore, though we will not stop to prove it, the free product operation we have defined on Cayley graphs is associative (and commutative as well, of course) so it makes sense to talk about the free product of any finite number of Cayley graphs, and, indeed, of any number of general rooted graphs. The following set of recursive definitions presents a generalization of our definition of the free product of two Cayley graphs.

Let $\{(\Gamma_i, e_i) : i = 1, \ldots, k\}$ be rooted graphs.

Definition 4.7 *The branch B_i is the graph formed by gluing copies of every B_j, $j \neq i$, by the root vertices, to each vertex of Γ_i except e_i. The free product $\Gamma_1 \star \Gamma_2 \star \cdots \star \Gamma_k$ is the graph*

$$B_1 \sharp B_2 \sharp \cdots \sharp B_k.$$

Again, we have an alternate form of this definition. Examining B_i as a subgraph of the full free product, we find that its root vertex lives in a copy of Γ_i which has copies of all the other B_j's (that is, all those with $j \neq i$) glued on at each non-root vertex (by the definition of B_i) and copies of all the other B_j's glued on at the root as well, from the connected-sum procedure. Thus the following definition is equivalent to the one above:

Definition 4.8 *The free product $\Gamma_1 \star \Gamma_2 \star \cdots \star \Gamma_k$ is the graph formed by gluing copies of every B_j, $j \neq i$, by the root vertices, to each vertex of Γ_i.*

4.4 Return generating function formulas

We are now ready to prove our main theorem.

Theorem 4.9 *Let $\{(\Gamma_i, e_i) : i = 1, \ldots, k\}$ be rooted graphs. Let $R_{e_i}(z)$ be the return generating function for Γ_i at e_i, and let $S_{e_i}(z)$ be the first-return generating function for the branch B_i in the free product*

$$\Gamma = \Gamma_1 \star \Gamma_2 \star \cdots \star \Gamma_k.$$

Let $R_e(z)$ be the return generating function for Γ at the root e. Then

1.

$$R_e(z) = \frac{1}{1 - \sum_i S_{e_i}(z)}$$

2. For each i,

$$R_e(z) = \frac{1}{1 - \sum_{j \neq i} S_{e_j}(z)} R_{e_i}\left(\frac{z}{1 - \sum_{j \neq i} S_{e_j}(z)}\right).$$

Proof. The first assertion is simply an application of theorem 2.7, since Γ is, by definition, the connected sum of its branches.

To prove the second assertion, we view Γ, as in definition 4.8, as a rooted copy of Γ_i with a bouquet of all the branches B_j, $j \neq i$, glued on at each vertex, including the root. If we focus our attention on this central copy of Γ_i, then a closed walk at e looks as follows. Starting at the vertex e, it takes some number of steps (possibly zero) in Γ_i, then leaves Γ_i, going into the bouquet glued to some vertex x. Because the bouquets are all independent and the closed walk has to get back to e, we know it must come back to x before taking its next step in Γ_i. That is, it completes a closed walk at the vertex x in the bouquet before continuing its journey through Γ_i.

Thus, every closed walk at e in Γ can be described as a closed walk at e in the central copy of Γ_i, with some number of side trips into the bouquets glued to all of Γ_i's vertices. Furthermore, since the root vertex e in Γ corresponds to the root e_i in Γ_i, and the bouquets of other branches are glued to Γ_i by *their* root vertices, we have the following one-to-one correspondences:

$$\left(\begin{array}{c}\text{Closed walks at } e\\ \text{in the central copy}\\ \text{of } \Gamma_i\end{array}\right) \longleftrightarrow \left(\begin{array}{c}\text{Closed walks at } e_i\\ \text{in } \Gamma_i\end{array}\right)$$

$$\left(\begin{array}{c}\text{Side trips into}\\ \text{a bouquet}\end{array}\right) \longleftrightarrow \left(\begin{array}{c}\text{Closed walks in}\\ \sharp_{j\neq i} B_j\end{array}\right)$$

We consider the following schematic description of a closed walk at e:

$$W_1\epsilon_1 W_2\epsilon_2 W_3\epsilon_3 \cdots W_p\epsilon_p W_{p+1}.$$

Each ϵ_j stands for an edge in the central copy of Γ_i, and the sequence $\epsilon_1\epsilon_2\cdots\epsilon_p$ describes a closed walk at e of length p, which always stays inside the central copy of Γ_i. Each W_j stands for a side trip (possibly of length zero) into the bouquet of branches glued to the j^{th} vertex visited by the closed walk $\epsilon_1\epsilon_2\cdots\epsilon_p$.

Because of the correspondences noted above, though, we can also interpret the sequence $\epsilon_1\epsilon_2\cdots\epsilon_p$ as a closed walk at e_i in Γ_i itself, and each W_j as a closed walk at the root of the connected sum of branches $\sharp_{j\neq i} B_j$. This makes it easier to count the number of such schematic representations. In fact, writing ρ_n for the coefficient of z^n in the full return generating function $R_e(z)$, we have

$$\rho_n \quad = \quad \sum_{p=0}^{n} \begin{pmatrix} \text{number of closed walks} \\ \epsilon_1\epsilon_2\cdots\epsilon_p \\ \text{at } e_i \text{ in } \Gamma_i \end{pmatrix} \begin{pmatrix} \text{number of sequences} \\ W_1, W_2, \ldots, W_{p+1} \\ \text{totalling } n-p \text{ steps} \end{pmatrix} \tag{13}$$

The closed walks in the first factor are counted by the return generating function $R_{e_i}(z)$. To help count the closed walks in the second factor, let us introduce the notation

$$T^{(i)}(z) \quad \stackrel{\text{def}}{=} \quad \frac{1}{1-\sum_{j\neq i} S_{e_j}(z)}. \tag{14}$$

Then, by theorem 2.7, $T^{(i)}(z)$ is the return generating function for the bouquet $\sharp_{j\neq i} B_j$. If we write

$$T^{(i)}(z) = \sum_{n=0}^{\infty} \tau_n^{(i)} z^n,$$

then $\tau_n^{(i)}$ is the number of possible closed “bouquet walks” W of length n.

In order to count sequences of W_j, we raise $T^{(i)}(z)$ to the k^{th} power:

$$(T^{(i)}(z))^k = \left(\sum_{n=0}^{\infty} \tau_n^{(i)} z^n\right)^k = \sum_{n=0}^{\infty} \left(\sum_{\gamma_1+\gamma_2+\cdots+\gamma_k=n} \tau_{\gamma_1}^{(i)}\tau_{\gamma_2}^{(i)}\cdots\tau_{\gamma_k}^{(i)} \right) z^n.$$

The coefficient of z^n in $(T^{(i)}(z))^k$ is thus seen to be the sum of the product $\tau_{\gamma_1}^{(i)}\tau_{\gamma_2}^{(i)}\cdots\tau_{\gamma_k}^{(i)}$, taken over all length-$k$ ordered partitions $\gamma_1+\gamma_2+\cdots+\gamma_k$ of n.

Side trips or bouquet walks are independent of one another, so by the multiplication principle, $\tau^{(i)}_{\gamma_1}\tau^{(i)}_{\gamma_2}\cdots\tau^{(i)}_{\gamma_k}$ is the number of sequences $W_1, W_2, \ldots, W_k$ of closed bouquet walks whose lengths are γ_1, γ_2, ..., and γ_k, respectively. The coefficient of z^n in $(T^{(i)}(z))^k$, which is the sum of this product over all length-k ordered partitions of n, is therefore equal to the total number of sequences $W_1, W_2, \ldots, W_k$ of closed bouquet walks such that the sum of the lengths of the walks in the sequence is n. The number of sequences $W_1, W_2, \ldots, W_{p+1}$ totalling $n-p$ steps, which appears in equation (13), is thus the coefficient of z^{n-p} in the expansion of $(T^{(i)}(z))^{p+1}$.

Using the notation $\rho^{(i)}_n$ for the coefficient of z^n in the return generating function $R_{e_i}(z)$, and recalling that $\rho^{(i)}_n$ is equal to the number of closed walks of length n at e_i in Γ_i, we can now rewrite expression (13) in a somewhat less wordy fashion:

$$\rho_n \quad = \quad \sum_{p=0}^{n} \rho^{(i)}_p (\text{coefficient of } z^{n-p} \text{ in } (T^{(i)}(z))^{p+1}). \tag{15}$$

Next we work from the other direction and expand $T^{(i)}(z)\, R_{e_i}(z\, T^{(i)}(z))$. We have

$$\begin{aligned} T^{(i)}(z)\, R_{e_i}(z\, T^{(i)}(z)) \quad &= \quad T^{(i)}(z) \sum_{p=0}^{\infty} \rho^{(i)}_p z^p\, (T^{(i)}(z))^p \\ &= \quad \sum_{p=0}^{\infty} \rho^{(i)}_p z^p\, (T^{(i)}(z))^{p+1}. \end{aligned}$$

For each p between 0 and n, the z^n term in this product gets a contribution equal to $\rho^{(i)}_p$ times the coefficient of z^{n-p} in the expansion of $(T^{(i)}(z))^{p+1}$. Thus the coefficient of z^n in $T^{(i)}(z)\, R_{e_i}(z\, T^{(i)}(z))$ is equal to

$$\sum_{p=0}^{n} \rho^{(i)}_p (\text{coefficient of } z^{n-p} \text{ in } (T^{(i)}(z))^{p+1}).$$

This matches our earlier expression for ρ_n, and allows us to conclude that the return generating function $R_e(z)$ is equal to $T^{(i)}(z)\, R_{e_i}(z\, T^{(i)}(z))$. Substituting $\dfrac{1}{1-\displaystyle\sum_{i\neq j} S_j(z)}$ back in for $T^{(i)}(z)$ gives us just what we want:

$$R_e(z) = \frac{1}{1 - \sum_{j \neq i} S_{e_j}(z)} R_{e_i} \left(\frac{z}{1 - \sum_{j \neq i} S_{e_j}(z)} \right).$$

□

If k factor graphs are involved in a free product, this theorem gives $k+1$ relations among the $2k+1$ quantities

$$R_{e_1}(z), R_{e_2}(z), \ldots, R_{e_k}(z), S_{e_1}(z), S_{e_2}(z), \ldots, S_{e_k}(z), R_e(z).$$

If, for each factor graph (Γ_i, e_i), we can find, by some other means, an expression for $R_{e_i}(z)$, then we have only $k+1$ unknown functions, and exactly as many relations among them. In theory, then, we can solve this system for the return generating function $R_e(z)$.

Unfortunately, in all but the simplest examples, the algebra can become quite daunting. In the next section, we present some of the simplest examples.

4.5 Examples II

The k-tree is the unique simply-connected graph with exactly k edges incident at every vertex. Figure 7 shows a portion of the 3-tree. The k-tree is clearly symmetric, and is subject to analysis using the tools in this paper, because it is a free product. In fact, the k-tree is the product

$$\underbrace{F_1 \star F_1 \star \cdots \star F_1}_{k \text{ factors}}.$$

We know that the return generating function for F_1 is

$$\frac{1}{1-z^2},$$

so by theorem 4.9, the return generating function $R_e(z)$ for the k-tree satisfies

$$R_e(z) = \frac{1}{1 - kS(z)} \tag{16}$$

and

$$R_e(z) = \frac{1}{1-(k-1)S(z)} \left[\frac{1}{1 - \left(\frac{z}{1-(k-1)S(z)} \right)^2} \right] \tag{17}$$

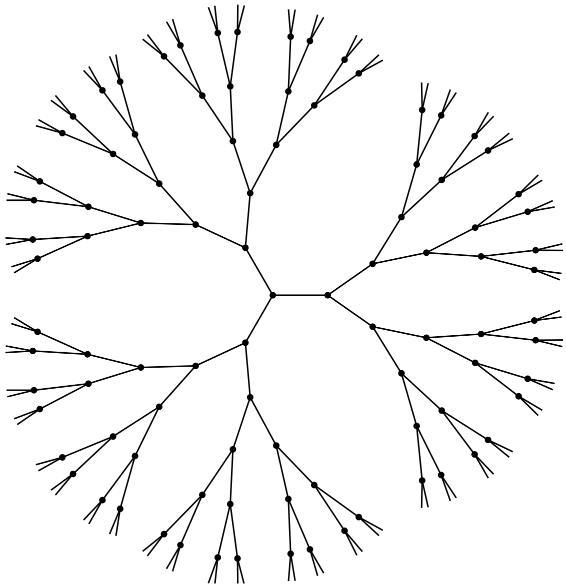


Figure 7: A portion of the 3-tree

where $S(z)$ is the (as yet unknown) first return generating function for a single branch of the k-tree. We can solve (16) and (17) for $S(z)$ and $R_e(z)$. The solutions are

$$S(z) = \frac{1-\sqrt{1-4z^2(k-1)}}{2(k-1)} \tag{18}$$

$$R_e(z) = \frac{2(k-1)}{k-2+k\sqrt{1-4z^2(k-1)}} \tag{19}$$

See [5] for a discussion of the coefficients in this generating function.

For a second example which yields to this computational technique, we consider the free product of two forks F_n and F_m. This is a simply-connected, non-symmetric graph with $n+m$ edges incident at the root vertex, and either $n+1$ or $m+1$ vertices incident at every other vertex. Figure 8 shows a portion of the graph $F_3 \star F_2$, with the root vertex at the center of the figure. The return generating functions for F_n and F_m are

$$\frac{1}{1-nz^2} \quad \text{and} \quad \frac{1}{1-mz^2}$$

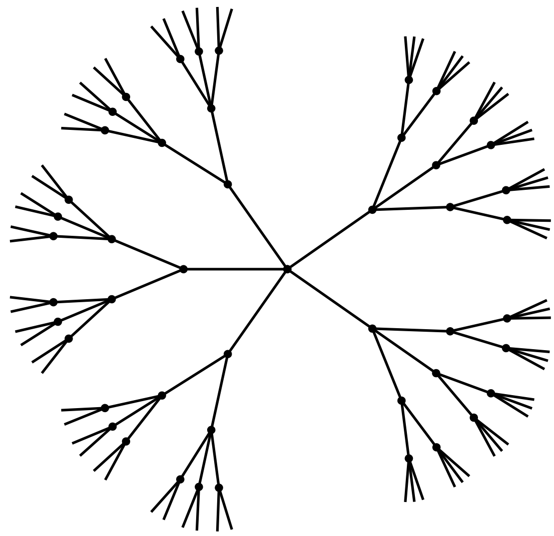


Figure 8: A portion of the graph $F_3 \star F_2$

respectively, so theorem 4.9 gives us the following set of equations:

$$\begin{aligned}
R_e(z) &= \frac{1}{1 - S_n(z) - S_m(z)} \\
R_e(z) &= \frac{1}{1 - S_n(z)} \left[\frac{1}{1 - m\left(\frac{z}{1-S_n(z)}\right)^2} \right] \\
R_e(z) &= \frac{1}{1 - S_m(z)} \left[\frac{1}{1 - n\left(\frac{z}{1-S_m(z)}\right)^2} \right]
\end{aligned}$$

This system has the solution

$$\begin{aligned}
S_n(z) &= \frac{1 + (n-m)z^2 - \sqrt{(m-n)^2 z^4 - 2(m+n)z^2 + 1}}{2} \\
S_m(z) &= \frac{1 + (m-n)z^2 - \sqrt{(m-n)^2 z^4 - 2(m+n)z^2 + 1}}{2} \\
R_e(z) &= \frac{1}{\sqrt{(m-n)^2 z^4 - 2(m+n)z^2 + 1}}.
\end{aligned}$$

Our computer tells us that the first few terms of the Taylor expansion of this generating function are

$$\begin{aligned}&1+\\&(m+n)z^2+\\&(m^2+4mn+n^2)z^4+\\&(m^3+9m^2n+9mn^2+n^3)z^6+\\&(m^4+16m^3n+36m^2n^2+16mn^3+n^4)z^8+\cdots.\end{aligned}$$

It would be an interesting challenge to explain combinatorially the appearance of the squares of the binomial coefficients in this expansion, if indeed that is the pattern.

For our last example, consider the graph $F_1 \star C_3$, part of which is shown in figure 9.

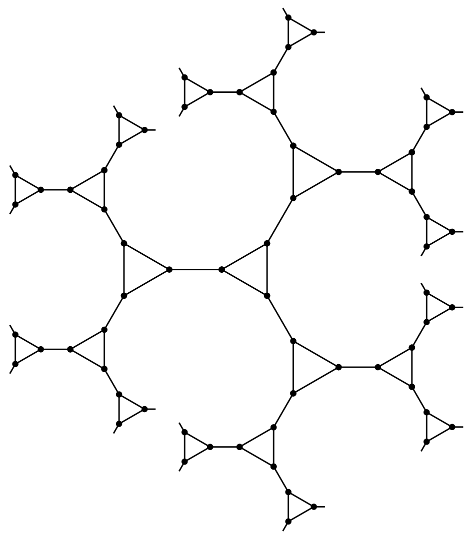


Figure 9: A portion of the graph $F_1 \star C_3$

We already know the return generating functions for the two factor graphs. For F_1, we have

$$R_e(z) = \frac{1}{1-z^2}$$

and for C_3 we have

$$R_e(z) = \frac{z-1}{2z^2+z-1}.$$

We will denote by $S_2(z)$ and $S_3(z)$ the first return generating functions for the branches of the free product graph. Theorem 4.9 gives us three expressions for the return generating function $R_e(z)$ of the free product graph:

$$
\begin{aligned}
R_e(z) &= \frac{1}{1-S_2(z)-S_3(z)} \\
R_e(z) &= \frac{1}{1-S_2(z)}\left[\frac{\frac{z}{1-S_2(z)}-1}{2\left(\frac{z}{1-S_2(z)}\right)^2+\frac{z}{1-S_2(z)}-1}\right] \\
R_e(z) &= \frac{1}{1-S_3(z)}\left[\frac{1}{1-\left(\frac{z}{1-S_3(z)}\right)^2}\right]
\end{aligned}
$$

Eliminating $R_e(z)$ for the moment, and suppressing the argument z, we get the relations

$$
\begin{aligned}
S_2(1-S_3) &= z^2 \\
S_3(1-S_2)-zS_3 &= 2z^2
\end{aligned}
$$

This system is quadratic in S_2 or S_3, and the solution is

$$
\begin{aligned}
S_2(z) &= -\frac{z^2+z-1+\sqrt{z^4+6z^3-5z^2-2z+1}}{2} \\
S_3(z) &= -\frac{z^2-z+1-\sqrt{z^4+6z^3-5z^2-2z+1}}{2(z-1)} \\
R_e(z) &= \frac{2(z-1)}{z^3+z^2-z+(z-2)\sqrt{z^4+6z^3-5z^2-2z+1}}
\end{aligned}
$$

Our computer says that the first eleven coefficients in the power series of $R_e(z)$ (for $n=0$ through $n=10$) are 1, 0, 3, 2, 15, 20, 89, 168, 591, 1346, and 4223.

5 Spectral results

In this section, we mention briefly how the return generating function is used in spectral theory. We will not be rigorous about this; we include this discussion just to demonstrate the usefulness of the return generating function.

In the present section, we will need our return generating functions to behave like functions and converge at least on some neighborhood of 0. They do; for a proof of this fact and other technical matters, we refer the reader to [4], from which we also quote the following important connection between the return generating function and the spectrum of an infinite graph. If the graph is symmetric, then our spectral expression for the return generating function,

$$R_e(z) = \frac{1}{N}\sum_i \frac{1}{1-\lambda_i z},$$

generalizes from the finite to the infinite case as

$$R_e(z) = \int \frac{d\mu(\lambda)}{1-\lambda z},$$

where $d\mu$ is the spectral measure associated with the operator A. The function of a new variable w, given by

$$\frac{1}{w}R_e\left(\frac{1}{w}\right) = \int \frac{d\mu(\lambda)}{w-\lambda}$$

is then the trace of the operator $(wI-A)^{-1}$. But the trace of $(wI-A)^{-1}$ is the Green's function $G(w)$ for the operator A, which contains complete information about the spectrum of the graph. Loosely speaking, the spectrum consists of the closure of the set of all numbers w for which $G(w)$ is not a real number.

On a symmetric graph, then, knowing $R_e(z)$, is equivalent to knowing $G(w)$, from which we can find out all we want to know about the graph's spectrum. We can use our earlier computation of the return generating function for the k-tree, for example, to verify the well-known fact (see [1]) that the spectrum of the k-tree is the interval

$$[-2\sqrt{k-1}, 2\sqrt{k-1}].$$

The Green's function for the k-tree is

$$\begin{aligned} G(w) &= \frac{1}{w}R_e\left(\frac{1}{w}\right) \\ &= \frac{2(k-1)}{w(k-2)+\operatorname{sgn}(w)k\sqrt{w^2-4(k-1)}}, \end{aligned}$$

which fails to produce a real number exactly when $|w| < 2\sqrt{k-1}$.

We conclude by repeating McLaughlin's calculation ([4], page 26) of the spectrum of the graph $F_1 \star C_3$ (see figure 9). From the return generating function

$$R_e(z) \;=\; \frac{2(z-1)}{z^3+z^2-z+(z-2)\sqrt{z^4+6z^3-5z^2-2z+1}},$$

we find the Green's function to be

$$G(w) \;=\; \frac{2(w-w^2)}{1+w-w^2+(1-2w)\sqrt{1+6w-5w^2-2w^3+w^4}}.$$

The expression under the radical sign is negative in the two intervals

$$\left(\frac{1-\sqrt{13+8\sqrt{2}}}{2}, \frac{1-\sqrt{13-8\sqrt{2}}}{2}\right)$$
$$\text{and}$$
$$\left(\frac{1+\sqrt{13-8\sqrt{2}}}{2}, \frac{1+\sqrt{13+8\sqrt{2}}}{2}\right)$$

and the entire denominator of $G(w)$ is 0 at $w = -2$ and (changing the sign of the radical) at $w = 0$. Thus the spectrum of $F_1 \star C_3$ is the union of the closures of the two intervals with the two points -2 and 0.

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